

A GAP FOR PPT ENTANGLEMENT

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ABSTRACT. Let W be a finite dimensional vector space over a field with characteristic not equal to 2. Denote by $\text{Sym}(V)$ and $\text{Skew-Sym}(V)$ the subspaces of symmetric and skew-symmetric tensors of a subspace V of $W \otimes W$, respectively. In this paper we show that if V is generated by tensors with tensor rank 1, $V = \text{Sym}(V) \oplus \text{Skew-Sym}(V)$ and W is the smallest vector space such that $V \subset W \otimes W$ then $\dim(\text{Sym}(V)) \geq \max\{\frac{2\dim(\text{Skew-Sym}(V))}{\dim(W)}, \frac{\dim(W)}{2}\}$.

This result has a straightforward application to the separability problem in Quantum Information Theory: If $\rho \in M_k \otimes M_k \simeq M_{k^2}$ is separable then $\text{rank}(Id + F)\rho(Id + F) \geq \max\{\frac{2}{r}\text{rank}(Id - F)\rho(Id - F), \frac{r}{2}\}$, where $F \in M_k \otimes M_k$ is the flip operator, $Id \in M_k \otimes M_k$ is the identity and r is the marginal rank of $\rho + F\rho F$. We prove the sharpness of this inequality.

Moreover, we show that if $\rho \in M_k \otimes M_k$ is positive under partial transposition (PPT) and $\text{rank}(Id + F)\rho(Id + F) = 1$ then ρ is separable. This result follows from Perron-Frobenius theory. We also present a large family of PPT matrices satisfying $\text{rank}(Id + F)\rho(Id + F) \geq r \geq \frac{2}{r-1}\text{rank}(Id - F)\rho(Id - F)$.

There is a possibility that an entangled PPT matrix $\rho \in M_k \otimes M_k$ satisfying $1 < \text{rank}(Id + F)\rho(Id + F) < \frac{2}{r}\text{rank}(Id - F)\rho(Id - F)$ exists. However, the family referenced above shows that finding one shall not be trivial.

INTRODUCTION

Let W be a finite dimensional vector space over a field with characteristic not equal to 2. Let V be a subspace of $W \otimes W$ and denote by $\text{Sym}(V)$ and $\text{Skew-Sym}(V)$ the subspaces of symmetric and skew-symmetric tensors of V , respectively.

If $V = \text{Sym}(V) \oplus \text{Skew-Sym}(V)$ and V is generated by tensors with tensor rank 1 then $\dim(\text{Sym}(V)) \neq 0$, since the tensor rank of every skew-symmetric tensor is not 1. Thus, we can ask the following question:

How small can the $\dim(\text{Sym}(V))$ be compared with $\dim(\text{Skew-Sym}(V))$, if V is generated by tensors with tensor rank 1 and $V = \text{Sym}(V) \oplus \text{Skew-Sym}(V)$?

This question is quite interesting for Quantum Information Theory. Let us identify $M_k \otimes M_k$ with M_{k^2} and $\mathbb{C}^k \otimes \mathbb{C}^k$ with \mathbb{C}^{k^2} via Kronecker product, where M_n is the set of complex matrices of order n .

One of the main problems in Quantum Information theory is discovering whether a positive semidefinite Hermitian matrix $\rho \in M_k \otimes M_k \simeq M_{k^2}$ is separable or not (see definition 2.1). Several necessary conditions for separability are known ([1–5]). One of these conditions is the so-called range criterion ([2]), i.e., the range (or the image) of a separable matrix $\rho \in M_k \otimes M_k \simeq M_{k^2}$ must be generated by tensors with tensor rank 1.

Observe that if $\rho \in M_k \otimes M_k$ is separable then the range of $2(\rho + F\rho F) = (Id + F)\rho(Id + F) + (Id - F)\rho(Id - F)$ has the same properties of V in the previous question, where $F \in M_k \otimes M_k$ is the flip operator (see definition 1.1). Thus, a solution for the previous question provides a necessary condition for the separability of ρ .

Here, we show that $\dim(\text{Sym}(V)) \geq \frac{2}{\dim(W)} \dim(\text{Skew-Sym}(V))$, if $V \subset W \otimes W$, $V = \text{Sym}(V) \oplus \text{Skew-Sym}(V)$ and V is generated by tensors with tensor rank 1 (theorem 1.5). For every W , we

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give an example of V such that $\dim(\text{Sym}(V)) = \frac{2}{\dim(W)} \dim(\text{Skew-Sym}(V))$ satisfying these two conditions (theorem 1.6). Moreover, if W is the smallest vector space such that $V \subset W \otimes W$ then $\dim(\text{Sym}(V)) \geq \frac{\dim(W)}{2}$. Therefore, $\dim(\text{Sym}(V)) \geq \max\{\frac{2}{\dim(W)} \dim(\text{Skew-Sym}(V)), \frac{\dim(W)}{2}\}$ (theorem 1.7).

Let $\rho \in M_k \otimes M_k$ and r denote the marginal rank of $\rho + F\rho F$ (see definition 2.2).

The inequality referenced above implies the following necessary condition for separability: If the range of a positive semidefinite Hermitian matrix ρ is generated by tensors with tensor rank 1 then $\text{rank}(Id + F)\rho(Id + F) \geq \max\{\frac{2}{r} \text{rank}(Id - F)\rho(Id - F), \frac{r}{2}\}$ (theorem 2.4 and definition 2.2). We prove the sharpness of this inequality (corollary 2.7).

Usually the range criterion is used when the range of a matrix does not contain tensors with tensor rank 1 ([6]). This inequality provides a very easy way to construct matrices whose range contains tensors with tensor rank 1, but is not generated by them (example 2.6).

Another necessary condition for the separability of ρ is to be positive under partial transposition ([1]). We can wonder if this inequality holds for matrices that are positive under partial transposition (PPT matrices). We are only able to prove this inequality for PPT matrices ρ such that marginal rank of $\rho + F\rho F$ is smaller or equal to 3 (corollary 3.6), but we obtain some partial results, which are of independent interest.

Firstly, we prove that if ρ is positive under partial transposition and $\text{rank}(Id + F)\rho(Id + F) = 1$ then ρ is separable (theorem 3.5). The proof of this theorem is quite technical, and requires a theorem from the Perron-Frobenius theory and some properties of the realignment map.

One possible approach to show that $\text{rank}(Id + F)\rho(Id + F) \geq \max\{\frac{2}{r} \text{rank}(Id - F)\rho(Id - F), \frac{r}{2}\}$ for a PPT matrix ρ is to find a lower bound for the rank $(Id + F)\rho(Id + F)$. For example, we know that the marginal ranks of a PPT matrix (definition 2.2) are lower bounds for its rank ([7, Theorem 1]). Unfortunately, $(Id + F)\rho(Id + F)$ does not need to be PPT, if ρ is PPT. Nevertheless we can impose some natural conditions on ρ , in order to obtain the PPT property for $(Id + F)\rho(Id + F)$.

Notice that the range of $(Id + F)\rho(Id + F)$ is a subspace of the symmetric tensors of $\mathbb{C}^k \otimes \mathbb{C}^k$. In order to be PPT, this matrix must have a symmetric Schmidt decomposition with positive coefficients (see [8, Section 3]). Here, we follow the nomenclature of [9–11] and we denote these matrices that have symmetric Schmidt decomposition with positive coefficients by symmetric with positive coefficients, or simply SPC matrices (definition 4.2). These papers showed that SPC matrices have strong connections with PPT matrices even if their ranges are not subspaces of the symmetric tensors.

Now, if we assume that ρ is PPT and SPC then $(Id + F)\rho(Id + F)$ is PPT and $\text{rank}(Id + F)\rho(Id + F) \geq r \geq \frac{2}{r-1} \text{rank}(Id - F)\rho(Id - F)$ (theorem 4.4 and corollary 4.6). Thus, there are plenty of non-trivial examples of PPT matrices satisfying $\text{rank}(Id + F)\rho(Id + F) \geq \frac{2}{r} \text{rank}(Id - F)\rho(Id - F)$.

Finally, since we don't know if this inequality holds for PPT matrices, there is a possibility that a PPT matrix ρ satisfying $1 < \text{rank}(Id + F)\rho(Id + F) < \frac{2}{r} \text{rank}(Id - F)\rho(Id - F)$ exists. In this case ρ is PPT and not separable. So this is a gap where we can look for PPT entanglement.

This paper is organized as follows: In Section 1, we prove that if a subspace V of $W \otimes W$ satisfies $V = \text{Sym}(V) \oplus \text{Skew-Sym}(V)$ and V is generated by tensors with tensor rank 1 then $\dim(\text{Sym}(V)) \geq \frac{2}{\dim(W)} \dim(\text{Skew-Sym}(V))$ (theorem 1.5). We also show that this inequality is sharp (theorem 1.6). Moreover, if W is the smallest vector space such that $V \subset W \otimes W$ then $\dim(\text{Sym}(V)) \geq \max\{\frac{2}{\dim(W)} \dim(\text{Skew-Sym}(V)), \frac{\dim(W)}{2}\}$.

In Section 2, we show that if $\rho \in M_k \otimes M_k \simeq M_{k^2}$ is separable then $\text{rank}(Id + F)\rho(Id + F) \geq \max\{\frac{2}{r} \text{rank}(Id - F)\rho(Id - F), \frac{r}{2}\}$. This inequality is also sharp (corollary 2.7).

In Section 3, we prove that ρ is separable, if ρ is positive under partial transposition and $\text{rank}((Id + F)\rho(Id + F)) = 1$ (theorem 3.5). We also show that if r is smaller or equal to 3 and ρ is PPT then $\text{rank}((Id + F)\rho(Id + F)) \geq \frac{2}{r}\text{rank}((Id - F)\rho(Id - F))$ (corollary 3.6).

In Section 4, we show that $(Id + F)\rho(Id + F)$ is PPT if ρ is PPT and $\rho + F\rho F$ is SPC. Under these conditions, we show that $\text{rank}((Id + F)\rho(Id + F)) \geq r \geq \frac{2}{r-1}\text{rank}((Id - F)\rho(Id - F))$ (theorem 4.4).

1. MAIN RESULTS

Let us begin this section with the following definition:

Definition 1.1. *Let W be a finite dimensional vector space over a field with characteristic not equal to 2.*

- (1) *Let $F : W \otimes W \rightarrow W \otimes W$ be the flip operator, i.e., $F(\sum_i a_i \otimes b_i) = \sum_i b_i \otimes a_i$. If $W = \mathbb{C}^k$ then $F = \sum_{i,j=1}^k e_i e_j^t \otimes e_j e_i^t \in M_k \otimes M_k$, where $\{e_1, \dots, e_k\}$ is the canonical basis of \mathbb{C}^k .*
- (2) *Let $M \subset W \otimes W$ and define $\text{Skew-Sym}(M) = \{w \in M \mid F(w) = -w\}$, $\text{Sym}(M) = \{w \in M \mid F(w) = w\}$.*
- (3) *Let $v \otimes M = \{v \otimes m_1, m_1 \in M\}$, $M \otimes v = \{m_2 \otimes v, m_2 \in M\}$ and $v \otimes M + M \otimes v = \{v \otimes m_1 + m_2 \otimes v \mid m_1, m_2 \in M\}$. Notice that $\text{Skew-Sym}(v \otimes W + W \otimes v) = \{v \otimes w - w \otimes v \mid w \in W\}$.*

In this section, we show that if a subspace V of $W \otimes W$ is invariant under flip operator (i.e, $V = \text{Sym}(V) \oplus \text{Skew-Sym}(V)$), and generated by tensors with tensor rank 1, then $\dim(\text{Sym}(V)) \geq \frac{2}{\dim(W)} \dim(\text{Skew-Sym}(V))$ (theorem 1.5). We also show that this inequality is sharp (theorem 1.6). Moreover, if W is the smallest vector space such that $V \subset W \otimes W$ then $\dim(\text{Sym}(V)) \geq \max\{\frac{2}{\dim(W)} \dim(\text{Skew-Sym}(V)), \frac{\dim(W)}{2}\}$ (theorem 1.7). In the next section, we provide applications to Quantum Information Theory.

In order to obtain our main theorem, we need the following two lemmas:

Lemma 1.2. *Let V be a subspace of $W \otimes W$, where W is a finite dimensional vector space over a field with characteristic not equal to 2. Let us assume that $F(V) \subset V$ and V has a generating subset formed by tensors with tensor rank 1. If there is $0 \neq w_1 \in W$ such that $\text{Skew-Sym}(w_1 \otimes W + W \otimes w_1) \subset V$ then $\dim(\text{Sym}(V)) \geq \dim(W) - 1$.*

Proof. Since $F(V) \subset V$, then $V = \text{Sym}(V) \oplus \text{Skew-Sym}(V)$. If $\dim(\text{Sym}(V)) = 0$, then every element of V is skew-symmetric. Therefore, the tensor rank of every element of V would not be 1, which is absurd. Thus, $\dim(\text{Sym}(V)) \geq 1$. So if $\dim(W) = 2$ then $\dim(\text{Sym}(V)) \geq 2 - 1$ and the result follows.

By induction, let us assume that this lemma is true for $\dim(W) \leq n - 1$. Let $\dim(W) = n > 2$.

Since $\dim(W) > 2$ then $\{0\} \neq \text{Skew-Sym}(w_1 \otimes W + W \otimes w_1)$.

There is $0 \neq a \otimes b \in V$ such that a or b is not a multiple of w_1 , otherwise $V = \text{span}\{w_1 \otimes w_1\}$ and $\text{Skew-Sym}(w_1 \otimes W + W \otimes w_1)$ would not be a subset of V . Since $b \otimes a = F(a \otimes b) \in V$, we may assume that a is not a multiple of w_1 .

Let $P : W \rightarrow W$ be a linear transformation such that $\text{rank}(P) = n - 1$, $P(a) = 0$ and $Pw_1 \neq 0$. Let $P \otimes P : W \otimes W \rightarrow W \otimes W$ be the linear transformation such that $P \otimes P(v \otimes w) = Pv \otimes Pw$.

Now, $P \otimes P(V) \subset P(W) \otimes P(W)$, $\dim(P(W)) = n - 1$ and since V is generated by tensors with tensor rank 1, then $P \otimes P(V)$ is also generated by tensors with tensor rank 1.

Next, notice that $F(P \otimes P(V)) = F(P \otimes P)F(F(V)) = P \otimes P(F(V)) \subset P \otimes P(V)$ and $\text{Skew-Sym}(Pw_1 \otimes P(W) + P(W) \otimes Pw_1) \subset P \otimes P(\text{Skew-Sym}(w_1 \otimes W + W \otimes w_1)) \subset P \otimes P(V)$. Thus, by induction hypothesis, $\dim(\text{Sym}(P \otimes P(V))) \geq (n - 1) - 1 = n - 2$.

Finally, $0 \neq a \otimes b + b \otimes a \in \text{Sym}(V) \cap \ker(P \otimes P)$ and since $P \otimes P(\text{Sym}(V)) \subset \text{Sym}(P \otimes P(V))$, $P \otimes P(\text{Skew-Sym}(V)) \subset \text{Skew-Sym}(P \otimes P(V))$, $V = \text{Sym}(V) \oplus \text{Skew-Sym}(V)$ then $P \otimes P :$

$\text{Sym}(V) \rightarrow \text{Sym}(P \otimes P(V))$ is surjective, therefore $\dim(\text{Sym}(V)) = \dim(\text{Sym}(P \otimes P(V))) + \dim(\ker(P \otimes P) \cap \text{Sym}(V)) \geq n - 2 + 1 = n - 1$. \square

Lemma 1.3. *Let V be a subspace of $W \otimes W$, where W is a finite dimensional vector space over a field \mathbb{K} with characteristic not equal to 2 and $F(V) \subset V$.*

Let G be a generating subset of V such that $F(G) = G$, the tensor rank of every element of G is 1 and $\text{span}\{v \mid v \otimes w \in G\} = W$. Moreover, assume that there exists $w_1 \otimes w_2 \in G$ such that if $c \otimes d \in G$ then $0 \neq w_1 \otimes c - c \otimes w_1 \in V$ or there exists $w_c \in W$ such that $0 \neq w_1 \otimes w_c - w_c \otimes w_1 \in V$ and $0 \neq c \otimes w_c - w_c \otimes c \in V$.

Then, $\dim(\text{Sym}(V)) \geq \dim(W) - 1$.

Proof. Since $F(V) \subset V$, then $V = \text{Sym}(V) \oplus \text{Skew-Sym}(V)$. If $\dim(\text{Sym}(V)) = 0$, then every element of V is skew-symmetric. Therefore, the tensor rank of every element of V would not be 1, which is absurd. Thus, $\dim(\text{Sym}(V)) \geq 1$. If $\dim(W) = 2$ then $\dim(\text{Sym}(V)) \geq 2 - 1$ and the result follows.

By induction, let us assume that this lemma is true if $\dim(W) \leq n - 1$ and let $\dim(W) = n > 2$.

There is $e \otimes f \in G$ such that $e \notin \text{span}\{w_1\}$, since $\text{span}\{v \mid v \otimes w \in G\} = W$, then $0 \neq w_1 \otimes e - e \otimes w_1 \in V$ or there exists $w_e \in W$ such that $0 \neq w_1 \otimes w_e - w_e \otimes w_1 \in V$. So $\text{Skew-Sym}(w_1 \otimes W + W \otimes w_1) \cap V \neq \{0\}$.

If $\text{Skew-Sym}(w_1 \otimes W + W \otimes w_1) \subset V$ then $\dim(\text{Sym}(V)) \geq n - 1$, by lemma 1.2. Let us assume that $\text{Skew-Sym}(w_1 \otimes W + W \otimes w_1)$ is not contained in V .

Thus, $\text{Skew-Sym}(w_1 \otimes W + W \otimes w_1) \cap V = \text{span}\{w_1 \otimes m_1 - m_1 \otimes w_1, \dots, w_1 \otimes m_l - m_l \otimes w_1\}$ and $\text{span}\{w_1, m_1, \dots, m_l\} \neq W$. Thus, there is $a \otimes b_1 \in G$ such that $a \notin \text{span}\{w_1, m_1, \dots, m_l\}$.

Let $P : W \rightarrow W$ be a linear transformation such that $P|_{\text{span}\{w_1, m_1, \dots, m_l\}} \equiv \text{Id}$ and $\ker(P) = \text{span}\{a\}$. Consider also the linear transformation $P \otimes P : W \otimes W \rightarrow W \otimes W$ and notice also that if $z \in \text{Skew-Sym}(w_1 \otimes W + W \otimes w_1) \cap V$ then $P \otimes P(z) = z$.

Let $V' = P \otimes P(V)$, $G' = \{Pv \otimes Pw \mid Pv \otimes Pw \neq 0 \text{ and } v \otimes w \in G\}$ and $W' = \text{span}\{P(v) \mid Pv \otimes Pw \in G'\}$. Notice that $\text{Skew-Sym}(w_1 \otimes W + W \otimes w_1) \cap V \subset V'$, since $P \otimes P(z) = z$ for every $z \in \text{Skew-Sym}(w_1 \otimes W + W \otimes w_1) \cap V$.

Notice that G' is a generating set of $V' = P \otimes P(V)$, $F(G') = G'$ and $V' \subset W' \otimes W'$. Recall that W' is a subset of the image of P then $\dim(W') \leq n - 1$. Now, $F(V') = F(P \otimes P)(V) = (P \otimes P)F(V) \subset (P \otimes P)(V) = V'$. Notice also the tensor rank of every element of G' is 1.

In order to complete this proof, we must show that

- (1) G' satisfies the last property of G in the hypothesis of this theorem and
- (2) if $(W \otimes a) \cap V = \{w \otimes a \mid w \in W \text{ and } w \otimes a \in V\}$ has dimension s then $\dim(W') \geq n - s$.

Therefore, by induction hypothesis, $\dim(\text{Sym}(P \otimes P(V))) \geq \dim(W') - 1$.

Since $P \otimes P(\text{Sym}(V)) \subset \text{Sym}(P \otimes P(V))$, $P \otimes P(\text{Skew-Sym}(V)) \subset \text{Skew-Sym}(P \otimes P(V))$, $V = \text{Sym}(V) \oplus \text{Skew-Sym}(V)$ then $P \otimes P : \text{Sym}(V) \rightarrow \text{Sym}(P \otimes P(V))$ is surjective. Let $\{b_1 \otimes a, \dots, b_s \otimes a\}$ be a basis of $(W \otimes a) \cap V$. Notice that $\{b_1 \otimes a + a \otimes b_1, \dots, b_s \otimes a + a \otimes b_s\}$ is a linear independent set and $\{b_1 \otimes a + a \otimes b_1, \dots, b_s \otimes a + a \otimes b_s\} \subset \ker(P \otimes P) \cap \text{Sym}(V)$. Finally, $\dim(\text{Sym}(V)) = \dim(\ker(P \otimes P) \cap \text{Sym}(V)) + \dim(\text{Sym}(P \otimes P(V))) \geq s + \dim(W') - 1 \geq s + n - s - 1 = n - 1$.

Proof of (1) :

Since $w_1 \otimes w_2 \in V$ then $w_1 \otimes w_2 - w_2 \otimes w_1 \in \text{Skew-Sym}(w_1 \otimes W + W \otimes w_1) \cap V$. Thus, $w_2 \in \text{span}\{w_1, m_1, \dots, m_l\}$ and $P(w_2) = w_2$. So $Pw_1 \otimes Pw_2 = w_1 \otimes w_2 \in G'$.

Now, let $Pc \otimes Pd \in G'$, where $c \otimes d \in G$. So $Pc \neq 0$, by definition of G' . If $Pw_1 \otimes Pc - Pc \otimes Pw_1 = 0$ then $Pc = \lambda Pw_1 = \lambda w_1$, $0 \neq \lambda \in \mathbb{K}$ (since $Pc \neq 0$). Since $0 \neq \text{Skew-Sym}(w_1 \otimes W + W \otimes w_1) \cap V \subset$

V' , there is $r \in W$ such that $0 \neq w_1 \otimes r - r \otimes w_1 \in V'$. Thus, $0 \neq \lambda(w_1 \otimes r - r \otimes w_1) = Pc \otimes r - r \otimes Pc \in V'$. Notice that $0 \neq w_1 \otimes r - r \otimes w_1 \in V' \subset W' \otimes W'$. Thus, $r \in W'$.

Next, if $0 \neq Pw_1 \otimes Pc - Pc \otimes Pw_1 \notin V' = P \otimes P(V)$ then $0 \neq w_1 \otimes c - c \otimes w_1 \notin V$. Thus, there exists $w_c \in W$ such that $0 \neq w_1 \otimes w_c - w_c \otimes w_1 \in V$ and $0 \neq c \otimes w_c - w_c \otimes c \in V$.

Notice that $0 \neq w_1 \otimes w_c - w_c \otimes w_1 = P \otimes P(w_1 \otimes w_c - w_c \otimes w_1) \in V' \subset W' \otimes W'$. Hence, $0 \neq Pw_c \in W'$.

Now, if $0 = Pc \otimes Pw_c - Pw_c \otimes Pc$ then $Pc = \mu Pw_c$, $0 \neq \mu \in \mathbb{K}$ (since $Pc \neq 0$ and $Pw_c \neq 0$). So $Pw_1 \otimes Pc - Pc \otimes Pw_1 = \mu(Pw_1 \otimes Pw_c - Pw_c \otimes Pw_1) \in V'$, which is a contradiction. Therefore, $0 \neq Pc \otimes Pw_c - Pw_c \otimes Pc \in V'$ (since $c \otimes w_c - w_c \otimes c \in V$) and $0 \neq w_1 \otimes Pw_c - Pw_c \otimes w_1 \in V'$ (since $w_1 \otimes w_c - w_c \otimes w_1 \in V$).

Thus, we have proved that there exists $w_1 \otimes w_2 \in G'$ such that if $Pc \otimes Pd \in G'$ then $0 \neq w_1 \otimes Pc - Pc \otimes w_1 \in V'$ or there exists $r \in W'$ such that $0 \neq w_1 \otimes r - r \otimes w_1 \in V'$ and $0 \neq Pc \otimes r - r \otimes Pc \in V'$. The proof of (1) is complete.

Proof of (2) :

Let $\{b_1 \otimes a, \dots, b_s \otimes a\}$ be a basis of $(W \otimes a) \cap V$ and recall that $a \otimes b_1 \in G$, so $b_1 \otimes a = F(a \otimes b_1) \in G$ too. Since $\text{span}\{v \mid v \otimes w \in G\} = W$ then there exists $\{b_{s+1} \otimes a_{s+1}, \dots, b_n \otimes a_n\} \subset G$ such that $\{b_1, \dots, b_n\}$ is basis for W .

Observe that if $v' \otimes w' \in W \otimes W$ is such that $Pv' \otimes Pw' = 0$ then $v' \in \text{span}\{a\}$ or $w' \in \text{span}\{a\}$. Moreover, if $v' \otimes w' \in G$ and $v' \in \text{span}\{a\}$ then $w' \otimes v' = F(v' \otimes w') \in (W \otimes a) \cap V$ and $w' \in \text{span}\{b_1, \dots, b_s\}$. Now, if $w' \in \text{span}\{a\}$ then $v' \otimes w' \in (W \otimes a) \cap V$ and $v' \in \text{span}\{b_1, \dots, b_s\}$. So if $Pv' \otimes Pw' = 0$ and $v' \otimes w' \in G$ then $v' \in \text{span}\{a\}$ and $w' \in \text{span}\{b_1, \dots, b_s\}$, or $v' \in \text{span}\{b_1, \dots, b_s\}$ and $w' \in \text{span}\{a\}$.

Next, if $P \otimes P(b_i \otimes a_i) = 0$, for some $i > s$, and $a_i \in \text{span}\{a\}$ then $b_i \in \text{span}\{b_1, \dots, b_s\}$, which is a contradiction (since $\{b_1, \dots, b_n\}$ is a basis). So if $P \otimes P(b_i \otimes a_i) = 0$, for $i > s$, then $b_i \in \text{span}\{a\}$.

Notice that if $a \notin \text{span}\{b_{s+1}, \dots, b_n\}$ and if $\sum_{i=s+1}^n \lambda_i Pb_i = 0$, for $\lambda_i \in \mathbb{K}$, then $\sum_{i=s+1}^n \lambda_i b_i \in \text{span}\{a\}$. Thus, $0 = \lambda_{s+1} = \dots = \lambda_n$ and $\{Pb_{s+1}, \dots, Pb_n\}$ is a linear independent set. In this case, every $b_i \notin \text{span}\{a\}$, for $i > s$, thus $Pb_i \otimes Pa_i \neq 0$. So $\{Pb_{s+1} \otimes Pa_{s+1}, \dots, Pb_n \otimes Pa_n\} \subset G'$, $\text{span}\{Pb_{s+1}, \dots, Pb_n\} \subset W'$ and $\dim(W') \geq n - s$.

Assume that $a \in \text{span}\{b_{s+1}, \dots, b_n\}$.

Let us prove that there is $p \otimes q \in G$ such that $Pp \otimes Pq \neq 0$ and $p \notin \text{span}\{b_{s+1}, \dots, b_n\}$ or $q \notin \text{span}\{b_{s+1}, \dots, b_n\}$. Since $q \otimes p = F(p \otimes q) \in G$ then we can assume $p \notin \text{span}\{b_{s+1}, \dots, b_n\}$. So $p = \sum_{i=1}^n \mu_i b_i$, $\mu_i \in \mathbb{K}$, and there exists $\mu_i \neq 0$ for some $i \leq s$. Without loss of generality assume $\mu_1 \neq 0$. Thus, $b_1 \in \text{span}\{p, b_2, \dots, b_n\}$ and $\{p, b_2, \dots, b_n\}$ is also a basis for W .

Since $\ker(P) \cap \text{span}\{p, b_{s+1}, \dots, b_n\} = \text{span}\{a\}$ then $\dim(\text{span}\{Pp, Pb_{s+1}, \dots, Pb_n\}) = \dim(\text{span}\{p, b_{s+1}, \dots, b_n\}) - \dim(\ker(P) \cap \text{span}\{p, b_{s+1}, \dots, b_n\}) = (n - s + 1) - 1 = n - s$.

Recall that, if $P \otimes P(b_i \otimes a_i) = 0$, for $i > s$, then $b_i \in \text{span}\{a\}$ and $Pb_i = 0 \in W'$. If $P \otimes P(b_i \otimes a_i) \neq 0$, for $i > s$, then $P \otimes P(b_i \otimes a_i) \in G'$ and $P(b_i) \in W'$. In any case, $\text{span}\{Pb_{s+1}, \dots, Pb_n\} \subset W'$. Recall that $Pp \in W'$, since $p \otimes q \in G$ and $Pp \otimes Pq \neq 0$. Thus, $\text{span}\{Pp, Pb_{s+1}, \dots, Pb_n\} \subset W'$ and $\dim(W') \geq n - s$.

Now, assume by contradiction that there is no $p \otimes q \in G$, such that $Pp \otimes Pq \neq 0$ and $p \notin \text{span}\{b_{s+1}, \dots, b_n\}$ or $q \notin \text{span}\{b_{s+1}, \dots, b_n\}$. So for every $p \otimes q \in G$ such that $Pp \otimes Pq \neq 0$, we have $\{p, q\} \subset \text{span}\{b_{s+1}, \dots, b_n\}$.

Notice that if $w_1 \otimes b_1 - b_1 \otimes w_1 = 0$ then $b_1 = \delta w_1$, $0 \neq \delta \in \mathbb{K}$, and $\delta(w_1 \otimes a - a \otimes w_1) = b_1 \otimes a - a \otimes b_1 \in V$. Since $P \otimes P(z) = z$ for every $z \in \text{Skew-Sym}(w_1 \otimes W + W \otimes w_1) \cap V$ then $b_1 \otimes a - a \otimes b_1 = \delta(w_1 \otimes a - a \otimes w_1) = P \otimes P(\delta(w_1 \otimes a - a \otimes w_1))$. Since $Pa = 0$ then $b_1 \otimes a - a \otimes b_1 = 0$ and $b_1 \in \text{span}\{a\} \subset \text{span}\{b_{s+1}, \dots, b_n\}$, which is a contradiction ($\{b_1, \dots, b_n\}$ is linear independent). Thus, $w_1 \otimes b_1 - b_1 \otimes w_1 \neq 0$.

Now, if $w_1 \otimes b_1 - b_1 \otimes w_1 \in V$ then we can write $w_1 \otimes b_1 - b_1 \otimes w_1 = \sum_j \lambda'_j v_j \otimes w_j + \sum_m \mu'_m c_m \otimes d_m$, where $\lambda'_j \in \mathbb{K}$, $\mu'_m \in \mathbb{K}$, $v_j \otimes w_j \in G \cap \ker(P \otimes P)$ and $c_m \otimes d_m \in G \setminus \ker(P \otimes P)$. By assumption, $\{c_m, d_m\} \subset \text{span}\{b_{s+1}, \dots, b_n\}$, for every m . Recall that, since $v_j \otimes w_j \in G \cap \ker(P \otimes P)$ then or $v_j \in \text{span}\{a\}$ and $w_j \in \text{span}\{b_1, \dots, b_s\}$, or $v_j \in \text{span}\{b_1, \dots, b_s\}$ and $w_j \in \text{span}\{a\}$.

Let $Q : W \rightarrow W$ be a linear transformation such that $Qb_i = 0$, for $1 \leq i \leq s$, and $Qb_i = b_i$, for $s+1 \leq i \leq n$. So $0 = Q \otimes Q(w_1 \otimes b_1 - b_1 \otimes w_1) = \sum_m \mu'_m c_m \otimes d_m$ and $w_1 \otimes b_1 - b_1 \otimes w_1 = \sum_j \lambda'_j v_j \otimes w_j$, where $v_j \in \text{span}\{a\}$ or $w_j \in \text{span}\{a\}$. So $0 \neq w_1 \otimes b_1 - b_1 \otimes w_1 = a \otimes r + s \otimes a$. Since $a \otimes r + s \otimes a$ is skew-symmetric then $s = -r$ and $0 \neq w_1 \otimes b_1 - b_1 \otimes w_1 = a \otimes r - r \otimes a$. Thus, $a = \lambda' w_1 + \lambda'' b_1$, where $\{\lambda', \lambda''\} \subset \mathbb{K}$.

If $\lambda'' = 0$ then $a = \lambda' w_1$ and $0 = P(a) = \lambda' P(w_1) = \lambda' w_1 = a$, which is a contradiction. So $0 \neq \lambda''(w_1 \otimes b_1 - b_1 \otimes w_1) = w_1 \otimes (\lambda' w_1 + \lambda'' b_1) - (\lambda' w_1 + \lambda'' b_1) \otimes w_1 = w_1 \otimes a - a \otimes w_1 \in V$.

Next, $0 = P \otimes P(w_1 \otimes a - a \otimes w_1) = w_1 \otimes a - a \otimes w_1$, since $P \otimes P(z) = z$ for every $z \in \text{Skew-Sym}(w_1 \otimes W + W \otimes w_1) \cap V$, which is a contradiction. Thus, $w_1 \otimes b_1 - b_1 \otimes w_1 \notin V$.

Since $w_1 \otimes b_1 - b_1 \otimes w_1 \notin V$ and $b_1 \otimes a \in G$ then, by the last property of G , there is $w_{b_1} \in W$ such that $0 \neq w_1 \otimes w_{b_1} - w_{b_1} \otimes w_1 \in V$ and $0 \neq b_1 \otimes w_{b_1} - w_{b_1} \otimes b_1 \in V$.

We can write $b_1 \otimes w_{b_1} - w_{b_1} \otimes b_1 = \sum_j \alpha_j v'_j \otimes w'_j + \sum_m \beta_m c'_m \otimes d'_m$, where $\{\alpha_j, \beta_m\} \subset \mathbb{K}$, $v'_j \otimes w'_j \in G \cap \ker(P \otimes P)$ and $c'_m \otimes d'_m \in G \setminus \ker(P \otimes P)$. We can repeat the argument above in order to obtain $a = \delta' w_{b_1} + \delta'' b_1$, where $\{\delta', \delta''\} \subset \mathbb{K}$.

If $\delta' = 0$ then $\delta'' b_1 = a \in \text{span}\{b_{s+1}, \dots, b_n\}$, which is a contradiction ($\{b_1, \dots, b_n\}$ is linear independent). If $\delta'' = 0$ then $a = \delta' w_{b_1}$ and $0 \neq \delta'(w_1 \otimes w_{b_1} - w_{b_1} \otimes w_1) = w_1 \otimes a - a \otimes w_1 \in V$, but $0 = P \otimes P(w_1 \otimes a - a \otimes w_1) = w_1 \otimes a - a \otimes w_1$. This is a contradiction.

Thus, $\delta' w_{b_1} = a - \delta'' b_1$ and $0 \neq \delta'(w_1 \otimes w_{b_1} - w_{b_1} \otimes w_1) = w_1 \otimes (a - \delta'' b_1) - (a - \delta'' b_1) \otimes w_1 \in V$.

We can write $w_1 \otimes (a - \delta'' b_1) - (a - \delta'' b_1) \otimes w_1 = \sum_j \epsilon_j v''_j \otimes w''_j + \sum_m \gamma_m c''_m \otimes d''_m$, where $\{\epsilon_j, \gamma_m\} \subset \mathbb{K}$, $v''_j \otimes w''_j \in G \cap \ker(P \otimes P)$ and $c''_m \otimes d''_m \in G \setminus \ker(P \otimes P)$.

Since $P \otimes P(z) = z$ for every $z \in \text{Skew-Sym}(w_1 \otimes W + W \otimes w_1) \cap V$ then $0 \neq w_1 \otimes (a - \delta'' b_1) - (a - \delta'' b_1) \otimes w_1 = P \otimes P(w_1 \otimes (a - \delta'' b_1) - (a - \delta'' b_1) \otimes w_1) = -\delta''(w_1 \otimes P b_1 - P b_1 \otimes w_1) = \sum_m \gamma_m P c''_m \otimes P d''_m$. Recall that $\{c''_m, d''_m\} \subset \text{span}\{b_{s+1}, \dots, b_n\}$, therefore $\{P c''_m, P d''_m\} \subset \text{span}\{P b_{s+1}, \dots, P b_n\}$.

Thus, $P b_1 \in \text{span}\{P b_{s+1}, \dots, P b_n\}$. We can write $P b_1 = \sum_{i=s+1}^n \zeta_i P b_i$, $\zeta_i \in \mathbb{K}$. Hence, $b_1 - \sum_{i=s+1}^n \zeta_i b_i \in \text{span}\{a\}$, and $b_1 \in \text{span}\{a, b_{s+1}, \dots, b_n\} = \text{span}\{b_{s+1}, \dots, b_n\}$, which is a contradiction. Therefore, there is $p \otimes q \in G$, such that $P p \otimes P q \neq 0$ and $p \notin \text{span}\{b_{s+1}, \dots, b_n\}$ or $q \notin \text{span}\{b_{s+1}, \dots, b_n\}$ and the proof is complete. \square

Corollary 1.4. *Let V be a subspace of $W \otimes W$, where W is a finite dimensional vector space over a field with characteristic not equal to 2. Let us assume that $F(V) \subset V$, V has a generating subset formed by tensors with tensor rank 1. If $\text{span}\{v \mid 0 \neq v \otimes w \in V\} = W$ and for every $0 \neq a' \otimes b' \in V$, we have $\dim(\text{Skew-Sym}(a' \otimes W + W \otimes a') \cap V) > \frac{\dim(W)}{2}$ then $\dim(\text{Sym}(V)) \geq \dim(W) - 1$.*

Proof. Let G be the set of all tensors in V with tensor rank 1. Notice that $F(G) = G$, G generates V and $\text{span}\{v \mid v \otimes w \in G\} = \text{span}\{v \mid 0 \neq v \otimes w \in V\} = W$. Let $w_1 \otimes w_2 \in G$ and $c \otimes d \in G$.

If $w_1 \otimes c - c \otimes w_1 = 0$ then $c = \lambda w_1$, $\lambda \neq 0$. Since $\dim(\text{Skew-Sym}(w_1 \otimes W + W \otimes w_1) \cap V) > \frac{\dim(W)}{2}$ then there is $0 \neq w_1 \otimes w'_c - w'_c \otimes w_1 \in V$. So there is $w'_c \in W$ such that $0 \neq c \otimes w'_c - w'_c \otimes c = \lambda(w_1 \otimes w'_c - w'_c \otimes w_1) \in V$.

Next, if $0 \neq w_1 \otimes c - c \otimes w_1 \notin V$ then let $\{w_1 \otimes m_1 - m_1 \otimes w_1, \dots, w_1 \otimes m_u - m_u \otimes w_1\}$ be a basis of $\text{Skew-Sym}(w_1 \otimes W + W \otimes w_1) \cap V$ and $\{c \otimes n_1 - n_1 \otimes c, \dots, c \otimes n_t - n_t \otimes c\}$ be a basis of $\text{Skew-Sym}(c \otimes W + W \otimes c) \cap V$. Notice that m_1, \dots, m_u are linear independent and n_1, \dots, n_t are linear independent. Notice that $w_1 \notin \text{span}\{m_1, \dots, m_u\}$, otherwise $\{w_1 \otimes m_1 - m_1 \otimes w_1, \dots, w_1 \otimes m_u - m_u \otimes w_1\}$ would not be linear independent. By the same reason $c \notin \text{span}\{n_1, \dots, n_t\}$.

By assumption $u > \frac{\dim(W)}{2}$ and $t > \frac{\dim(W)}{2}$. Thus, $\text{span}\{m_1, \dots, m_u\} \cap \text{span}\{n_1, \dots, n_t\} \neq \{0\}$. Let $0 \neq w_c \in \text{span}\{m_1, \dots, m_u\} \cap \text{span}\{n_1, \dots, n_t\}$. Notice that w_c and w_1 are linear independent

since $w_1 \notin \text{span}\{m_1, \dots, m_u\}$, so $0 \neq w_1 \otimes w_c - w_c \otimes w_1 \in V$. Analogously we obtain $0 \neq c \otimes w_c - w_c \otimes c \in V$.

Finally, G satisfies the hypothesis of lemma 1.3, therefore $\dim(\text{Sym}(V)) \geq \dim(W) - 1$. \square

Theorem 1.5. *Let V be a subspace of $W \otimes W$, where W is a finite dimensional vector space over a field with characteristic not equal to 2. Let us assume that $F(V) \subset V$, V has a generating subset formed by tensors with tensor rank 1. Then, $\dim(\text{Sym}(V)) \geq \frac{2}{\dim(W)} \dim(\text{Skew-Sym}(V))$.*

Proof. Since $F(V) \subset V$ then $V = \text{Sym}(V) \oplus \text{Skew-Sym}(V)$. If $\dim(\text{Sym}(V)) = 0$ then every element of V is skew-symmetric. Therefore the tensor rank of every element of V is not 1, which is absurd. Thus, $\dim(\text{Sym}(V)) \geq 1$. If $\dim(W) = 2$ then $1 \geq \dim(\text{Skew-Sym}(V))$ and $\dim(\text{Sym}(V)) \geq \frac{2}{2} \dim(\text{Skew-Sym}(V))$.

By induction, let us assume that this theorem is true when $2 \leq \dim(W) \leq n - 1$ and let $\dim(W) = n$.

Observe that if $R = \text{span}\{v \mid 0 \neq v \otimes w \in V\} \neq W$ then $\dim(R) \leq n - 1$. Since $F(V) \subset V$ and V is generated by tensors with tensor rank 1 then $V \subset R \otimes R$. By induction hypothesis, $\dim(\text{Sym}(V)) \geq \frac{2}{\dim(R)} \dim(\text{Skew-Sym}(V)) \geq \frac{2}{n} \dim(\text{Skew-Sym}(V))$.

Now, let us assume that $\text{span}\{v \mid 0 \neq v \otimes w \in V\} = W$.

If for every $0 \neq a' \otimes b' \in V$, we have $\dim(\text{Skew-Sym}(a' \otimes W + W \otimes a') \cap V) > \frac{n}{2}$ then $\dim(\text{Sym}(V)) \geq n - 1$, by corollary 1.4. Since $\dim(\text{Skew-Sym}(V)) \leq \frac{n(n-1)}{2}$ then $\dim(\text{Sym}(V)) \geq \frac{2}{n} \dim(\text{Skew-Sym}(V))$.

Next, let us assume that there is $0 \neq a \otimes b \in V$ such that $\dim(\text{Skew-Sym}(a \otimes W + W \otimes a) \cap V) \leq \frac{n}{2}$.

Let $P : W \rightarrow W$ be a linear transformation such that $\ker P = \text{span}\{a\}$. Recall that $a \otimes W = \{a \otimes w \mid w \in W\}$, $W \otimes a = \{w \otimes a \mid w \in W\}$ and $\ker(P \otimes P) = a \otimes W + W \otimes a$.

If $V \subset \ker(P \otimes P)$, since V is generated by tensors with tensor rank 1, then V is generated by $((a \otimes W) \cup (W \otimes a)) \cap V$. Moreover, since $F(V) \subset V$ then the linear transformations $P_1 : (a \otimes W) \cap V \rightarrow \text{Sym}(V)$, $P_1(a \otimes w) = a \otimes w + w \otimes a$, and $P_2 : (a \otimes W) \cap V \rightarrow \text{Skew-Sym}(V)$, $P_2(a \otimes w) = a \otimes w - w \otimes a$, are surjective. Note that P_1 is also injective, since the characteristic of \mathbb{K} is not 2. Thus, $\dim(\text{Sym}(V)) = \dim((a \otimes W) \cap V) \geq \dim(\text{Skew-Sym}(V)) > \frac{2}{n} \dim(\text{Skew-Sym}(V))$, since $n > 2$.

Next, assume that $0 \neq P \otimes P(V)$ and let $W' = \text{span}\{Pv \mid 0 \neq Pv \otimes Pw \text{ and } v \otimes w \in V\}$ and $s = \dim(W')$. Since $0 \neq P \otimes P(V)$ then $0 < s \leq \text{rank}(P) = n - 1$. Observe that $F(P \otimes P(V)) = P \otimes P(F(V)) \subset P \otimes P(V)$ and $P \otimes P(V)$ is generated by tensors with tensor rank 1. Therefore, $P \otimes P(V) \subset W' \otimes W'$.

Thus, $P \otimes P(V)$ satisfies the same conditions of V and by induction hypothesis, $\dim(\text{Sym}(P \otimes P(V))) \geq \frac{2}{s} \dim(\text{Skew-Sym}(P \otimes P(V)))$. Consider $P \otimes P : W \otimes W \rightarrow W \otimes W$.

Recall that $0 \neq a \otimes b + b \otimes a \in \ker(P \otimes P) \cap \text{Sym}(V)$ and $\text{Skew-Sym}(a \otimes W + W \otimes a) \cap V = \ker(P \otimes P) \cap \text{Skew-Sym}(V)$.

Notice that, since $P \otimes P(\text{Sym}(V)) \subset \text{Sym}(P \otimes P(V))$, $P \otimes P(\text{Skew-Sym}(V)) \subset \text{Skew-Sym}(P \otimes P(V))$ and $V = \text{Sym}(V) \oplus \text{Skew-Sym}(V)$ then $P \otimes P : \text{Sym}(V) \rightarrow \text{Sym}(P \otimes P(V))$ and $P \otimes P : \text{Skew-Sym}(V) \rightarrow \text{Skew-Sym}(P \otimes P(V))$ are surjective.

Since $\dim(\text{Sym}(V)) = \dim(\ker(P \otimes P) \cap \text{Sym}(V)) + \dim(\text{Sym}(P \otimes P(V)))$ then $\dim(\text{Sym}(V)) \geq 1 + \frac{2}{s} \dim(\text{Skew-Sym}(P \otimes P(V))) \geq 1 + \frac{2}{n} \dim(\text{Skew-Sym}(P \otimes P(V)))$.

Note that, $\dim(\text{Skew-Sym}(V)) = \dim(\ker(P \otimes P) \cap \text{Skew-Sym}(V)) + \dim(\text{Skew-Sym}(P \otimes P(V))) = \dim(\text{Skew-Sym}(a \otimes W + W \otimes a) \cap V) + \dim(\text{Skew-Sym}(P \otimes P(V))) \leq \frac{n}{2} + \dim(\text{Skew-Sym}(P \otimes P(V)))$. Thus, $1 + \frac{2}{n} \dim(\text{Skew-Sym}(P \otimes P(V))) \geq \frac{2}{n} \dim(\text{Skew-Sym}(V))$ and $\dim(\text{Sym}(V)) \geq \frac{2}{n} \dim(\text{Skew-Sym}(V))$. \square

Theorem 1.6. *Let W be a k -dimensional vector space over a field \mathbb{K} with characteristic not equal to 2. There is a subspace V of $W \otimes W$, such that $F(V) \subset V$, V has a generating subset*

formed by tensors with tensor rank 1, $\text{span}\{v \mid 0 \neq v \otimes w \in V\} = W$ and $\dim(\text{Sym}(V)) = \frac{2}{k} \dim(\text{Skew-Sym}(V))$. Thus, the inequality in theorem 1.5 is sharp.

Proof. Let w_1, \dots, w_k be a basis of W and let e_1, \dots, e_k be the canonical basis of \mathbb{K}^k . Let $G : W \rightarrow \mathbb{K}^k$ be the linear transformation such that $G(w)$ is the vector of the coordinates of w in the basis w_1, \dots, w_k .

Observe that $G \otimes G : W \otimes W \rightarrow \mathbb{K}^k \otimes \mathbb{K}^k$, $G \otimes G(\sum_i c_i \otimes d_i) = \sum_i G(c_i) \otimes G(d_i)$, is an isomorphism and the tensor rank of $m \in W \otimes W$ is the tensor rank of $G \otimes G(m) \in \mathbb{K}^k \otimes \mathbb{K}^k$. Notice also that $G \otimes G : \text{Sym}(W \otimes W) \rightarrow \text{Sym}(\mathbb{K}^k \otimes \mathbb{K}^k)$ and $G \otimes G : \text{Skew-Sym}(W \otimes W) \rightarrow \text{Skew-Sym}(\mathbb{K}^k \otimes \mathbb{K}^k)$ are also isomorphisms. Thus, if we can find a subspace V of $\mathbb{K}^k \otimes \mathbb{K}^k$ satisfying the required properties then $(G \otimes G)^{-1}(V)$ is a subspace of $W \otimes W$ satisfying the same properties. Now, let us construct this V inside $\mathbb{K}^k \otimes \mathbb{K}^k$.

Let $M_k(\mathbb{K})$ be the set of matrices of order k with coefficients in \mathbb{K} . Consider the linear transformation $T : \mathbb{K}^k \otimes \mathbb{K}^k \rightarrow M_k(\mathbb{K})$, $T(\sum_i f_i \otimes g_i) = \sum_i f_i g_i^t$. Observe that the tensor rank of $v \in \mathbb{K}^k \otimes \mathbb{K}^k$ is the rank of $T(v)$ and $T(F(v)) = T(v)^t$, where $F : \mathbb{K}^k \otimes \mathbb{K}^k \rightarrow \mathbb{K}^k \otimes \mathbb{K}^k$ is the flip operator (definition 1.1).

Let $W_i = \text{span}\{e_1, \dots, e_i\}$ and $a_2 = e_2 \otimes (e_1 + e_2)$, $a_3 = e_3 \otimes (e_1 + 2e_2 + e_3), \dots, a_k = e_k \otimes (e_1 + 2e_2 + \dots + 2e_{k-1} + e_k)$.

Define $V_i = \text{span}\{a_2, F(a_2), \dots, a_i, F(a_i)\} + \text{Skew-Sym}(W_i \otimes W_i)$. Notice that $F(V_i) \subset V_i$ and $\text{Sym}(V_i) = \text{span}\{a_2 + F(a_2), \dots, a_i + F(a_i)\}$. Notice that $\text{span}\{a_2 + F(a_2), \dots, a_{i-1} + F(a_{i-1})\} \subset W_{i-1} \otimes W_{i-1}$ and $a_i + F(a_i) \notin W_{i-1} \otimes W_{i-1}$, so $a_i + F(a_i) \notin \text{span}\{a_2 + F(a_2), \dots, a_{i-1} + F(a_{i-1})\}$, for every i . Thus, $\{a_2 + F(a_2), \dots, a_i + F(a_i)\}$ is a linear independent set and $\dim(\text{Sym}(V_i)) = i - 1 = \frac{2}{i} \dim(\text{Skew-Sym}(W_i \otimes W_i)) = \frac{2}{i} \dim(\text{Skew-Sym}(V_i))$. Notice also that $\text{span}\{v \mid 0 \neq v \otimes w \in V_i\} = \text{span}\{e_1 + e_2, e_2, \dots, e_i\} = W_i$, for $i \geq 2$.

In order to complete this proof, we must show, by induction on i , that each V_i has a generating subset formed by tensors with tensor rank 1, and then we choose $V = V_k$.

Notice that $\text{Skew-Sym}(W_2 \otimes W_2) = \text{span}\{e_1 \otimes e_2 - e_2 \otimes e_1\} \subset \text{span}\{a_2, F(a_2)\}$. So $\text{span}\{a_2, F(a_2)\}$ is a generating subset of V_2 . By induction, let us assume that V_{n-1} has a generating subset formed by tensors with tensor rank 1.

Let $s_i = a_i + F(a_i) + \dots + a_n + F(a_n)$ and $r_i = (e_1 + 2e_2 + \dots + 2e_{i-1} + e_i + \dots + e_n) \otimes (e_i + \dots + e_n)$, $i \geq 2$. Let us prove that $s_i = r_i + F(r_i)$. Notice that

$$T(s_i) = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 & \dots & 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 2 & \dots & 2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 2 & \dots & 2 & 0 & \dots & 0 \\ 1 & 2 & \dots & 2 & 2 & \dots & 2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & \dots & 2 & 2 & \dots & 2 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix}, T(r_i) = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 & \dots & 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 2 & \dots & 2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 2 & \dots & 2 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 1 & \dots & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 1 & \dots & 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix},$$

where the first $i - 1$ rows and columns of $T(s_i)$ are multiples of $(0, \dots, 0, 1, \dots, 1, 0, \dots, 0)$, the next $n - i + 1$ rows and columns of $T(s_i)$ are equal to $(1, 2, \dots, 2, \dots, 2, 0, \dots, 0)$ and the last $k - n$ rows and columns of $T(s_i)$ are zero. Notice that $T(s_i) = T(r_i) + T(r_i)^t = T(r_i + F(r_i))$. Thus, $s_i = r_i + F(r_i)$ and r_i has tensor rank 1 for $2 \leq i \leq n$.

Next, the n^{th} row of $T(r_2 - F(r_2)) = T(r_2) - T(r_2)^t$ is $(-1, 0, \dots, 0)$, the n^{th} row of $T(r_3 - F(r_3)) = T(r_3) - T(r_3)^t$ is $(-1, -2, 0, \dots, 0)$, \dots , the n^{th} row of $T(r_n - F(r_n)) = T(r_n) - T(r_n)^t$ is $(-1, -2, \dots, -2, 0, \dots, 0)$.

Hence, $\{A \in M_k(\mathbb{K}) \mid A = -A^t, a_{ij} = 0 \text{ if } i > n-1 \text{ or } j > n-1\} + \text{span}\{T(r_2 - F(r_2)), \dots, T(r_n - F(r_n))\} = \{A \in M_k(\mathbb{K}) \mid A = -A^t, a_{ij} = 0 \text{ if } i > n \text{ or } j > n\}$.

Since $T(\text{Skew-Sym}(W_s \otimes W_s)) = \{A \in M_k(\mathbb{K}) \mid A = -A^t, a_{ij} = 0 \text{ if } i > s \text{ or } j > s\}$ then $\text{Skew-Sym}(W_{n-1} \otimes W_{n-1}) + \text{span}\{r_2 - F(r_2), r_3 - F(r_3), \dots, r_n - F(r_n)\} = \text{Skew-Sym}(W_n \otimes W_n)$.

So $V_n = \text{span}\{a_2, F(a_2), \dots, a_n, F(a_n)\} + \text{Skew-Sym}(W_n \otimes W_n) = \text{span}\{a_2, F(a_2), \dots, a_n, F(a_n)\} + \text{span}\{r_2 - F(r_2), r_3 - F(r_3), \dots, r_n - F(r_n)\} + \text{Skew-Sym}(W_{n-1} \otimes W_{n-1})$.

Since $\text{span}\{r_2 + F(r_2), r_3 + F(r_3), \dots, r_n + F(r_n)\} \subset \text{span}\{a_2, F(a_2), \dots, a_{n-1}, F(a_{n-1}), a_n, F(a_n)\}$ then $\text{span}\{a_2, F(a_2), \dots, a_{n-1}, F(a_{n-1}), a_n, F(a_n)\} + \text{span}\{r_2 - F(r_2), r_3 - F(r_3), \dots, r_n - F(r_n)\} = \text{span}\{a_2, F(a_2), \dots, a_{n-1}, F(a_{n-1}), a_n, F(a_n), r_2, r_3, \dots, r_n\}$.

Hence, $V_n = \text{span}\{a_2, F(a_2), \dots, a_{n-1}, F(a_{n-1}), a_n, F(a_n), r_2, r_3, \dots, r_n\} + \text{Skew-Sym}(W_{n-1} \otimes W_{n-1})$.

Finally, $V_n = V_{n-1} + \text{span}\{a_n, F(a_n), r_2, r_3, \dots, r_n\}$. By induction hypothesis, V_{n-1} has a generating set formed by tensors with tensor rank 1 then V_n has a generating set formed by tensors with tensor rank 1. \square

We complete this section adding one assumption to theorem 1.5. We prove that if V satisfies the hypothesis of theorem 1.5 and $\text{span}\{v \mid 0 \neq v \otimes w \in V\} = W$ then $\dim(\text{Sym}(V)) \geq \max\{\frac{2 \dim(\text{Skew-Sym}(V))}{\dim(W)}, \frac{\dim(W)}{2}\}$. Moreover, we analyze both cases: $\dim(\text{Sym}(V)) = \frac{\dim(W)}{2}$ and $\dim(\text{Sym}(V)) = \frac{2 \dim(\text{Skew-Sym}(V))}{\dim(W)}$.

Theorem 1.7. *Let V be a subspace of $W \otimes W$, where W is a finite dimensional vector space over a field \mathbb{K} with characteristic not equal to 2. Let us assume that $F(V) \subset V$, V has a generating subset formed by tensors with tensor rank 1. If $\text{span}\{v \mid 0 \neq v \otimes w \in V\} = W$ then $\dim(\text{Sym}(V)) \geq \max\{\frac{2 \dim(\text{Skew-Sym}(V))}{\dim(W)}, \frac{\dim(W)}{2}\}$. Moreover,*

- a) *If $\dim(\text{Sym}(V)) = \frac{\dim(W)}{2}$ then $\dim(\text{Skew-Sym}(V)) = \dim(\text{Sym}(V))$.*
- b) *If $\dim(\text{Sym}(V)) = \frac{2 \dim(\text{Skew-Sym}(V))}{\dim(W)}$ then $\dim(\text{Sym}(V)) = \dim(W) - 1$ and $\text{Skew-Sym}(V) = \text{Skew-Sym}(W \otimes W)$.*

Proof. The inequality $\dim(\text{Sym}(V)) \geq \frac{2 \dim(\text{Skew-Sym}(V))}{\dim(W)}$ was proved in theorem 1.5.

Let $\{a_1 \otimes b_1, \dots, a_n \otimes b_n\}$ be a basis of V . Thus, $\{a_1 \otimes b_1 + b_1 \otimes a_1, \dots, a_n \otimes b_n + b_n \otimes a_n\}$ is a generating set of $\text{Sym}(V)$. Without loss of generality, assume $\{a_1 \otimes b_1 + b_1 \otimes a_1, \dots, a_t \otimes b_t + b_t \otimes a_t\}$ is a basis of $\text{Sym}(V)$.

Let $0 \neq v \otimes w \in V$ then $v \otimes w + w \otimes v \in \text{span}\{a_1 \otimes b_1 + b_1 \otimes a_1, \dots, a_t \otimes b_t + b_t \otimes a_t\}$. Therefore, $v \in \text{span}\{a_1, \dots, a_t, b_1, \dots, b_t\}$ and $W = \text{span}\{v \mid 0 \neq v \otimes w \in V\} \subset \text{span}\{a_1, \dots, a_t, b_1, \dots, b_t\}$. Thus, $\dim(W) \leq 2t = 2 \dim(\text{Sym}(V))$.

Now, let us prove item a).

Notice that if $\dim(W) = 2 \dim(\text{Sym}(V)) = 2t$ then $W = \text{span}\{a_1, \dots, a_t, b_1, \dots, b_t\}$ and the set $\{a_1, \dots, a_t, b_1, \dots, b_t\}$ is a basis of W .

Let $v \otimes w + w \otimes v = \sum_{j=1}^s \alpha_j (a_{i_j} \otimes b_{i_j} + b_{i_j} \otimes a_{i_j})$, where $\alpha_j \neq 0$ for every j .

Since $\{a_{i_1}, \dots, a_{i_s}, b_{i_1}, \dots, b_{i_s}\} \subset \{a_1, \dots, a_t, b_1, \dots, b_t\}$ then $\{a_{i_1}, \dots, a_{i_s}, b_{i_1}, \dots, b_{i_s}\}$ is a linear independent set and since $\alpha_j \neq 0$, for every j , the tensor rank of $\sum_{j=1}^s \alpha_j (a_{i_j} \otimes b_{i_j} + b_{i_j} \otimes a_{i_j})$ is $2s$. Since the tensor rank of $v \otimes w + w \otimes v$ is 1 or 2 then $s = 1$, the tensor rank of $v \otimes w + w \otimes v$ is 2 and $v \otimes w + w \otimes v = \alpha_1 (a_{i_1} \otimes b_{i_1} + b_{i_1} \otimes a_{i_1})$. Therefore, $\text{span}\{v, w\} = \text{span}\{a_{i_1}, b_{i_1}\}$.

So $\beta(a_{i_1} \otimes b_{i_1} - b_{i_1} \otimes a_{i_1}) = v \otimes w - w \otimes v$, for some $\beta \in \mathbb{K}$. Since V is generated by $\{v \otimes w \mid 0 \neq v \otimes w \in V\}$ and $v \otimes w - w \otimes v$ is equal to some $a_i \otimes b_i - b_i \otimes a_i$, $1 \leq i \leq t$, then

$\{a_1 \otimes b_1 - b_1 \otimes a_1, \dots, a_t \otimes b_t - b_t \otimes a_t\}$ is a generating set of $\text{Skew-Sym}(V)$. Since $\{a_1, \dots, a_t, b_1, \dots, b_t\}$ is a linear independent set then $\{a_1 \otimes b_1 - b_1 \otimes a_1, \dots, a_t \otimes b_t - b_t \otimes a_t\}$ is also a linear independent set. Therefore, $\dim(\text{Skew-Sym}(V)) = t = \dim(\text{Sym}(V))$.

Next, let us prove item b).

If $\dim(W) = 1$ then $\dim(\text{Skew-Sym}(V)) = \dim(\text{Skew-Sym}(W \otimes W)) = 0$. So $\frac{2 \dim(\text{Skew-Sym}(V))}{\dim(\text{Sym}(V))} = \dim(W)$ implies $\dim(W) \geq 2$ and $\dim(\text{Skew-Sym}(V)) \geq \dim(\text{Sym}(V))$.

Let $0 \neq a' \otimes b' \in V$ and $P : W \rightarrow W$ be a linear transformation such that $\ker(P) = \text{span}\{a'\}$. Denote by $a' \otimes W = \{a' \otimes w \mid w \in W\}$ and $W \otimes a' = \{w \otimes a' \mid w \in W\}$. Thus, $\ker(P \otimes P) = a' \otimes W + W \otimes a'$.

Assume $V \subset \ker(P \otimes P)$. Since V is generated by tensors with tensor rank 1 then V is generated by $((a' \otimes W) \cup (W \otimes a')) \cap V$. Thus, the linear transformations $P_1 : (a' \otimes W) \cap V \rightarrow \text{Sym}(V)$, $P_1(a' \otimes w) = a' \otimes w + w \otimes a'$, and $P_2 : (W \otimes a') \cap V \rightarrow \text{Skew-Sym}(V)$, $P_2(w \otimes a') = a' \otimes w - w \otimes a'$, are surjective. Note that P_1 is also injective, since the characteristic of \mathbb{K} is not 2. Thus, $\dim(\text{Sym}(V)) = \dim((a' \otimes W) \cap V) \geq \dim(\text{Skew-Sym}(V))$.

Therefore, $\dim(\text{Skew-Sym}(V)) = \dim(\text{Sym}(V))$ and $\dim(W) = \frac{2 \dim(\text{Skew-Sym}(V))}{\dim(\text{Sym}(V))} = 2$. Hence, $\dim(\text{Skew-Sym}(V)) = 1$, $\text{Skew-Sym}(V) = \text{Skew-Sym}(W \otimes W)$ and $\dim(\text{Sym}(V)) = 1 = \dim(W) - 1$.

Now, assume that $P \otimes P(V) \neq 0$ and notice that $\dim(\ker(P \otimes P) \cap \text{Sym}(V)) \geq 1$, since $0 \neq a' \otimes b' + b' \otimes a' \in \ker(P \otimes P) \cap \text{Sym}(V)$.

Since $P \otimes P(\text{Sym}(V)) \subset \text{Sym}(P \otimes P(V))$, $P \otimes P(\text{Skew-Sym}(V)) \subset \text{Skew-Sym}(P \otimes P(V))$, $V = \text{Sym}(V) \oplus \text{Skew-Sym}(V)$ then $P \otimes P : \text{Sym}(V) \rightarrow \text{Sym}(P \otimes P(V))$ and $P \otimes P : \text{Skew-Sym}(V) \rightarrow \text{Skew-Sym}(P \otimes P(V))$ are surjective, thus $\dim(\text{Sym}(V)) = \dim(\text{Sym}(P \otimes P(V))) + \dim(\ker(P \otimes P) \cap \text{Sym}(V))$ and $\dim(\text{Skew-Sym}(V)) = \dim(\text{Skew-Sym}(P \otimes P(V))) + \dim(\ker(P \otimes P) \cap \text{Skew-Sym}(V))$.

Next, since $0 \neq P \otimes P(V) \subset P(W) \otimes P(W)$, $P \otimes P(V)$ is generated by tensors with tensor rank 1 and is invariant under flip operator then $\dim(\text{Sym}(P \otimes P(V))) \geq 1$ and $\frac{2 \dim(\text{Skew-Sym}(P \otimes P(V)))}{\dim(\text{Sym}(P \otimes P(V)))} \leq \dim(P(W)) = \dim(W) - 1$, by theorem 1.5.

Note that if $\dim(\text{Skew-Sym}(V) \cap \ker(P \otimes P)) \leq \frac{\dim(W)}{2}$ then $\frac{2 \dim(\text{Skew-Sym}(V) \cap \ker(P \otimes P))}{\dim(\text{Sym}(V) \cap \ker(P \otimes P))} \leq \dim(W)$.

Since $\frac{2 \dim(\text{Skew-Sym}(V))}{\dim(\text{Sym}(V))}$ is a non-trivial convex combination of $\frac{2 \dim(\text{Skew-Sym}(V) \cap \ker(P \otimes P))}{\dim(\text{Sym}(V) \cap \ker(P \otimes P))}$ and $\frac{2 \dim(\text{Skew-Sym}(P \otimes P(V)))}{\dim(\text{Sym}(P \otimes P(V)))}$ ($\frac{\dim(\text{Sym}(V) \cap \ker(P \otimes P))}{\dim \text{Sym}(V)} + \frac{\dim(\text{Sym}(P \otimes P(V)))}{\dim \text{Sym}(V)} = 1$) then $\frac{2 \dim(\text{Skew-Sym}(V))}{\dim(\text{Sym}(V))} < \dim(W)$, if $\dim(\text{Skew-Sym}(V) \cap \ker(P \otimes P)) \leq \frac{\dim(W)}{2}$.

So $\frac{2 \dim(\text{Skew-Sym}(V))}{\dim(\text{Sym}(V))} = \dim(W)$ implies $\dim(\text{Skew-Sym}(V) \cap \ker(P \otimes P)) > \frac{\dim(W)}{2}$. Thus, for every $0 \neq a' \otimes b' \in V$, we have $\dim(\text{Skew-Sym}(V) \cap (a' \otimes W + W \otimes a')) = \dim(\text{Skew-Sym}(a' \otimes W + W \otimes a') \cap V) > \frac{\dim(W)}{2}$. Thus, by corollary 1.4, we have $\dim(\text{Sym}(V)) \geq \dim(W) - 1$. Finally, $\dim(\text{Skew-Sym}(W \otimes W)) \geq \dim(\text{Skew-Sym}(V)) = \frac{\dim(W) \dim(\text{Sym}(V))}{2} \geq \frac{\dim(W)(\dim(W)-1)}{2} = \dim(\text{Skew-Sym}(W \otimes W))$. Therefore, $\text{Skew-Sym}(W \otimes W) = \text{Skew-Sym}(V)$ and $\dim(\text{Sym}(V)) = \frac{2 \dim(\text{Skew-Sym}(V))}{\dim(W)} = \dim(W) - 1$. \square

2. APPLICATIONS TO QUANTUM INFORMATION THEORY

In this section, we show that if $\rho \in M_k \otimes M_k \simeq M_{k^2}$ is separable and r is the marginal rank of $\rho + F\rho F$ then $\text{rank}(Id + F)\rho(Id + F) \geq \max\{\frac{2}{r} \text{rank}(Id - F)\rho(Id - F), \frac{r}{2}\}$ (corollary 2.5). We also show that this inequality is sharp (corollary 2.7).

Let M_k denote the set of complex matrices of order k and \mathbb{C}^k be the set of column vectors with k complex entries. We shall identify the tensor product space $\mathbb{C}^k \otimes \mathbb{C}^m$ with \mathbb{C}^{km} and the tensor product space $M_k \otimes M_m$ with M_{km} , via Kronecker product (i.e., if $A = (a_{ij}) \in M_k$ and $B \in M_m$ then $A \otimes B = (a_{ij}B) \in M_{km}$. If $v = (v_i) \in \mathbb{C}^k$ and $w \in \mathbb{C}^m$ then $v \otimes w = (v_i w) \in \mathbb{C}^{km}$).

The identification of the tensor product space $\mathbb{C}^k \otimes \mathbb{C}^m$ with \mathbb{C}^{km} and the tensor product space $M_k \otimes M_m$ with M_{km} , via Kronecker product, allow us to write $(v \otimes w)(r \otimes s)^t = vr^t \otimes ws^t$, where $v \otimes w$ is a column, $(v \otimes w)^t$ its transpose and $v, r \in \mathbb{C}^k$ and $w, s \in \mathbb{C}^m$. Therefore if $x, y \in \mathbb{C}^k \otimes \mathbb{C}^m \simeq \mathbb{C}^{km}$ we have $xy^t \in M_k \otimes M_m \simeq M_{km}$.

The image (or the range) of the matrix $\rho \in M_k \otimes M_m \simeq M_{km}$ in $\mathbb{C}^k \otimes \mathbb{C}^m \simeq \mathbb{C}^{km}$ shall be denoted by $\mathfrak{S}(\rho)$.

Definition 2.1. (*Separable Matrices*) Let $\rho \in M_k \otimes M_m$. We say that ρ is separable if $\rho = \sum_{i=1}^n C_i \otimes D_i$ such that $C_i \in M_k$ and $D_i \in M_m$ are positive semidefinite Hermitian matrices for every i . If ρ is not separable then ρ is entangled.

Definition 2.2. Let $\rho = \sum_{i=1}^m A_i \otimes B_i \in M_k \otimes M_k$. Define $\rho^A = \sum_{i=1}^m A_i \text{tr}(B_i) \in M_k$ and $\rho^B = \sum_{i=1}^m B_i \text{tr}(A_i) \in M_k$. The matrices ρ^A, ρ^B are usually called the marginal or local matrices. The marginal ranks of ρ are the ranks of ρ^A and ρ^B . If they are equal, we shall call them the marginal rank of ρ .

Remark 2.3. It is well known that if $\rho \in M_k \otimes M_k \simeq M_{k^2}$ is a positive semidefinite Hermitian matrix then $\rho^A \in M_k$ and $\rho^B \in M_k$ are too. Moreover, $(F\rho F)^A = \rho^B$, $(F\rho F)^B = \rho^A$ and $\mathfrak{S}(\rho) \subset \mathfrak{S}(\rho^A) \otimes \mathfrak{S}(\rho^B)$.

Theorem 2.4. Let $\rho \in M_k \otimes M_k \simeq M_{k^2}$ be a positive semidefinite hermitian matrix. If $\mathfrak{S}(\rho)$ is generated by tensors with tensor rank 1 and r is the marginal rank of $\rho + F\rho F$ then

$$\text{rank} (Id + F)\rho(Id + F) \geq \max\{\frac{2}{r}\text{rank} (Id - F)\rho(Id - F), \frac{r}{2}\},$$

where $F \in M_k \otimes M_k$ is the flip operator, $Id \in M_k \otimes M_k$ is the identity.

Proof. Firstly, notice that $(\rho + F\rho F)^A = (\rho + F\rho F)^B$, and let us denote this marginal matrix by σ . By remark 2.3, $\mathfrak{S}(\rho + F\rho F) \subset \mathfrak{S}(\sigma) \otimes \mathfrak{S}(\sigma)$ and, by hypothesis, $\text{rank}(\sigma) = r$.

Secondly, notice that the range of $B = 2(\rho + F\rho F) = (Id + F)\rho(Id + F) + (Id - F)\rho(Id - F)$ is generated by tensors with tensor rank 1, is invariant under flip operator and is a subset of $\mathfrak{S}(\sigma) \otimes \mathfrak{S}(\sigma)$. Moreover, $\dim(\text{Sym}(\mathfrak{S}(B))) = \text{rank}(Id + F)\rho(Id + F)$ and $\dim(\text{Skew-Sym}(\mathfrak{S}(B))) = \text{rank}(Id - F)\rho(Id - F)$. Therefore, by theorem 1.5, $\text{rank}((Id + F)\rho(Id + F)) \geq \frac{2}{r}\text{rank}((Id - F)\rho(Id - F))$.

Now, let $W = \text{span}\{v \mid 0 \neq v \otimes w \in \mathfrak{S}(B)\}$. Since B is generated by tensors with tensor rank 1 and $\text{tr}(B(m\overline{m}^t \otimes Id)) = 2\text{tr}(\sigma m\overline{m}^t)$ then $m \in W^\perp$ if and only if $m \in \ker(\sigma)$. Thus, $W = \mathfrak{S}(\sigma)$. Finally, by theorem 1.7, $\text{rank}((Id + F)\rho(Id + F)) \geq \frac{r}{2}$. \square

Corollary 2.5. If $\rho \in M_k \otimes M_k \simeq M_{k^2}$ is separable and r is the marginal rank of $\rho + F\rho F$ then $\text{rank} (Id + F)\rho(Id + F) \geq \max\{\frac{2}{r}\text{rank} (Id - F)\rho(Id - F), \frac{r}{2}\}$.

Proof. By the range criterion [2], $\mathfrak{S}(\rho)$ has a generating subset formed by tensors with tensor rank 1. Now, use theorem 2.4. \square

Example 2.6. Let $B \in M_k \otimes M_k \simeq M_{k^2}$ be a positive semidefinite Hermitian matrix with rank smaller than $k - 1$. The matrix $\rho = B + (Id - F)$ is not separable since $\text{rank} (Id + F)\rho(Id + F) = \text{rank} (Id + F)B(Id + F) < k - 1 = \frac{2}{k}\text{rank} (Id - F)\rho(Id - F)$.

Corollary 2.7. For every k , there is a separable matrix $\rho \in M_k \otimes M_k$ such that the marginal rank of $\rho + F\rho F$ is k and $\text{rank} (Id + F)\rho(Id + F) = k - 1 = \frac{2}{k}\text{rank} (Id - F)\rho(Id - F)$. Therefore the inequality of theorem 2.4 is sharp.

Proof. Let $V \subset \mathbb{C}^k \otimes \mathbb{C}^k$ be the vector space described in theorem 1.6. Let $\{v_1 \otimes w_1, \dots, v_m \otimes w_m\}$ be a basis for V and consider $\rho = \sum_{i=1}^m v_i \overline{v_i}^t \otimes w_i \overline{w_i}^t$.

Notice that ρ is separable and $(Id + F)\rho(Id + F) = \sum_{i=1}^m s_i \overline{s_i}^t$, where $s_i = v_i \otimes w_i + w_i \otimes v_i$. Notice that $\{s_1, \dots, s_m\}$ is a generating set for $\text{Sym}(V)$ and for the image of $(Id + F)\rho(Id + F)$.

So $\text{rank}(Id + F)\rho(Id + F) = \dim(\text{Sym}(V))$. Analogously, we have $\text{rank}(Id - F)\rho(Id - F) = \dim(\text{Skew-Sym}(V))$. Thus, $\text{rank}(Id + F)\rho(Id + F) = \frac{2}{k}\text{rank}(Id - F)\rho(Id - F)$.

Notice that $\frac{2}{k}\text{rank}(Id - F)\rho(Id - F) = \text{rank}(Id + F)\rho(Id + F) \geq \frac{2}{r}\text{rank}(Id - F)\rho(Id - F)$, where r is the marginal rank of $\rho + F\rho F$. Thus, $r \geq k$. Since $\rho + F\rho F \in M_k \otimes M_k$ then $r \leq k$. Therefore, $r = k$. Finally, by item b) of theorem 1.7, we have $\dim(\text{Sym}(V)) = k - 1 = \text{rank}(Id + F)\rho(Id + F)$ \square

3. A GAP FOR PPT ENTANGLEMENT

In this section we prove that if $\rho \in M_k \otimes M_k \simeq M_{k^2}$ is positive under partial transposition (definition 3.1) and $\text{rank}((Id + F)\rho(Id + F)) = 1$ then ρ is separable (theorem 3.5).

We saw in theorem 2.4 that if $\rho \in M_k \otimes M_k$ is separable then $\text{rank}(Id + F)\rho(Id + F) \geq \frac{2}{r}\text{rank}(Id - F)\rho(Id - F)$, where r is the marginal rank of $\rho + F\rho F$.

Notice that there is a possibility that a PPT matrix ρ satisfying $1 < \text{rank}(Id + F)\rho(Id + F) < \frac{2}{r}\text{rank}(Id - F)\rho(Id - F)$ exists. In this case ρ is entangled. So this is a gap where we can look for PPT entanglement.

Here, we also prove that $\text{rank}((Id + F)\rho(Id + F)) \geq \frac{2}{r}\text{rank}((Id - F)\rho(Id - F))$ for any PPT matrix $\rho \in M_k \otimes M_k$, when $r \leq 3$ (corollary 3.6). In the next section, we provide several non-trivial examples of PPT matrices $\rho \in M_k \otimes M_k \simeq M_{k^2}$ such that $\text{rank}(Id + F)\rho(Id + F) \geq r \geq \frac{2}{r-1}\text{rank}(Id - F)\rho(Id - F)$.

We shall denote by A^{t_2} the matrix $\sum_{i=1}^n A_i \otimes B_i^t$, which is called the partial transposition of $A = \sum_{i=1}^n A_i \otimes B_i \in M_k \otimes M_m \simeq M_{km}$.

Definition 3.1. (PPT matrices) Let $A = \sum_{i=1}^n A_i \otimes B_i \in M_k \otimes M_m \simeq M_{km}$ be a positive semidefinite Hermitian matrix. We say that A is positive under partial transposition or simply PPT, if $A^{t_2} = Id \otimes (\cdot)^t(A) = \sum_{i=1}^n A_i \otimes B_i^t$ is positive semidefinite.

Definition 3.2. Let $V : M_k \rightarrow \mathbb{C}^k \otimes \mathbb{C}^k$ be defined by $V(\sum_{i=1}^n a_i b_i^t) = \sum_{i=1}^n a_i \otimes b_i$ and let $R : M_k \otimes M_k \rightarrow M_k \otimes M_k$ be defined by $R(\sum_{i=1}^n A_i \otimes B_i) = \sum_{i=1}^n V(A_i)V(B_i)^t$, where $V(A_i) \in \mathbb{C}^k \otimes \mathbb{C}^k$ is a column vector and $V(B_i)^t$ is a row vector. This map $R : M_k \otimes M_k \rightarrow M_k \otimes M_k$ is usually called the “realignment map” (See [3–5]).

Lemma 3.3. (Properties of the Realignment map) Let $R : M_n \otimes M_n \rightarrow M_n \otimes M_n$ be the realignment map of definition 3.2 and $F \in M_n \otimes M_n$ the flip operator of definition 1.1. Let $v_i, w_i \in \mathbb{C}^n \otimes \mathbb{C}^n$ and $C \in M_n \otimes M_n$. Then,

- (1) $R(\sum_{i=1}^m v_i w_i^t) = \sum_{i=1}^m V^{-1}(v_i) \otimes V^{-1}(w_i)$
- (2) $R(CF)F = C^{t_2}$
- (3) $R(CF) = R(C)^{t_2}$
- (4) $R(C^{t_2}) = R(C)F$
- (5) $R(C^{t_2}) = (CF)^{t_2}$

Proof. See [11, Lemma 23] for the items 1 to 4. For the last item, notice that it is sufficient to prove this formula for $C = ab^t \otimes cd^t$, where $\{a, b, c, d\} \subset \mathbb{C}^k$, and the proof is straightforward. \square

Remark 3.4. Let $u = \sum_{i=1}^n e_i \otimes e_i \in \mathbb{C}^n \otimes \mathbb{C}^n$, where $\{e_1, \dots, e_n\}$ is the canonical basis of \mathbb{C}^n . Observe that $Id = \sum_{i,j=1}^n e_i e_i^t \otimes e_j e_j^t$, $uu^t = \sum_{i,j=1}^n e_i e_j^t \otimes e_i e_j^t$, $R(Id) = \sum_{i,j=1}^n V(e_i e_i^t)V(e_j e_j^t)^t = \sum_{i,j=1}^k e_i e_j^t \otimes e_i e_j^t = uu^t$ and $R(uu^t) = \sum_{i,j=1}^n V(e_i e_j^t)V(e_i e_j^t)^t = \sum_{i,j=1}^k e_i e_i^t \otimes e_j e_j^t = Id$.

Theorem 3.5. Let $\rho \in M_k \otimes M_k \simeq M_{k^2}$ be a positive semidefinite hermitian matrix, $Id \in M_k \otimes M_k \simeq M_{k^2}$ the identity and $F \in M_k \otimes M_k$ the flip operator. Suppose the rank of $(Id + F)\rho(Id + F)$ is 1. If ρ is positive under partial transposition then the marginal rank of $\rho + F\rho F$ is smaller or equal to 2 and ρ is separable.

Proof. Let $(Id + F)\rho(Id + F) = w\bar{w}^t$ and let us prove that the tensor rank of w is smaller or equal to 2.

Now, if ρ is PPT then $\rho + F\rho F$ is also PPT. Notice that $B = 2(\rho + F\rho F) = (Id + F)\rho(Id + F) + (Id - F)\rho(Id - F) = w\bar{w}^t + \sum_{j=1}^m b_j \bar{b}_j^t$, where $r \in \text{Sym}(\mathbb{C}^k \otimes \mathbb{C}^k)$ and $b_j \in \text{Skew-Sym}(\mathbb{C}^k \otimes \mathbb{C}^k)$, $1 \leq j \leq m$.

Let n be the tensor rank of w . Since $w \in \text{Sym}(\mathbb{C}^k \otimes \mathbb{C}^k)$ then there are linear independent vectors s_1, \dots, s_n in \mathbb{C}^k such that $w = \sum_{i=1}^n s_i \otimes s_i$.

Let $T \in M_{n \times k}(\mathbb{C})$ be such that $Ts_i = e_i$, where e_1, \dots, e_n is the canonical basis of \mathbb{C}^n . Notice that $C = (T \otimes T)B(T^* \otimes T^*) \in M_n \otimes M_n$ is also PPT and $C = uu^t + \sum_{j=1}^m a_j \bar{a}_j^t$, where $u = (T \otimes T)r = \sum_{i=1}^n e_i \otimes e_i$ and $a_j = (T \otimes T)b_j \in \text{Skew-Sym}(\mathbb{C}^n \otimes \mathbb{C}^n)$, $1 \leq j \leq m$.

Let $R : M_n \otimes M_n \rightarrow M_n \otimes M_n$ be the realignment map (definition 3.2). Now, $C^{t_2} = (R(C)^{t_2})F$, by properties 2 and 3 in lemma 3.3.

Now, $R(C) = Id + \sum_{j=1}^m A_j \otimes \bar{A}_j \in M_n \otimes M_n$, where $Id = R(uu^t)$ and $A_j \otimes \bar{A}_j = V^{-1}(a_j) \otimes V^{-1}(\bar{a}_j)$, by remark 3.4 and by property 1 in lemma 3.3. Notice that each $A_j = V^{-1}(a_j)$ is a complex skew-symmetric matrix, since $a_j \in \text{Skew-Sym}(\mathbb{C}^n \otimes \mathbb{C}^n)$. Thus, $R(C)^{t_2} = Id - \sum_{j=1}^m A_j \otimes \bar{A}_j$.

Let $A_j = A'_j + iA''_j$, where A'_j, A''_j are real skew-symmetric matrices in M_n .

Thus, $A_j \otimes \bar{A}_j = A'_j \otimes A'_j + A''_j \otimes A''_j + i(A'_j \otimes A''_j - A''_j \otimes A'_j)$ and $C^{t_2} = (R(C)^{t_2})F = (Id - (\sum_{j=1}^m A'_j \otimes A'_j + A''_j \otimes A''_j))F - i(\sum_{j=1}^m A'_j \otimes A''_j - A''_j \otimes A'_j)F$. Notice that $C^{t_2} = P + iQ$, where $P = (Id - (\sum_{j=1}^m A'_j \otimes A'_j + A''_j \otimes A''_j))F$, $Q = -(\sum_{j=1}^m A'_j \otimes A''_j - A''_j \otimes A'_j)F$ are real matrices, because F is a real matrix.

Since C is PPT then C^{t_2} is a positive semidefinite Hermitian matrix then $P = (Id - (\sum_{j=1}^m A'_j \otimes A'_j + A''_j \otimes A''_j))F$ is a positive semidefinite symmetric matrix.

Next, notice that $L(X) = \sum_{j=1}^m A'_j X A_j^t + A''_j X A_j^{t_2}$ is a positive map acting on M_n , by theorem 2.5 in [12], there is a positive semidefinite Hermitian matrix Y , which is an eigenvector associated to the spectral radius of $L(X)$. Let $Y = S' + iA'$, where S' is a real symmetric matrix and A' is a real skew-symmetric matrix and notice that $S' \neq 0$. Notice that the sets of symmetric and skew-symmetric matrices are left invariant by $L(X)$. Thus, S' is also an eigenvector of $L(X)$ associated to the spectral radius.

Now, $V \circ L \circ V^{-1}(v) = (\sum_{j=1}^m A'_j \otimes A'_j + A''_j \otimes A''_j)v$, for every $v \in \mathbb{C}^k \otimes \mathbb{C}^k$, where V is defined in definition 3.2. Therefore, there exists a symmetric tensor $V(S') = s' \in \mathbb{C}^n \otimes \mathbb{C}^n$ such that $(\sum_{j=1}^m A'_j \otimes A'_j + A''_j \otimes A''_j)s' = \lambda s'$, where λ is the spectral radius of this matrix. Notice that $\sum_{j=1}^m A'_j \otimes A'_j + A''_j \otimes A''_j$ is a real symmetric matrix, so the spectral radius is the biggest eigenvalue.

So $Ps' = (Id - (\sum_{j=1}^m A'_j \otimes A'_j + A''_j \otimes A''_j))Fs' = (1 - \lambda)s'$. Thus, the biggest eigenvalue of $\sum_{j=1}^m A'_j \otimes A'_j + A''_j \otimes A''_j$ is smaller or equal to 1 and $PF = Id - (\sum_{j=1}^m A'_j \otimes A'_j + A''_j \otimes A''_j)$ is also positive semidefinite.

Since P and PF are positive semidefinite then $P = PF$. By properties 2 and 4 in lemma 3.3, we have $(PF)^{t_2} = P^{t_2} = R(PF)F = R((PF)^{t_2})$. Therefore, $Id + \sum_{j=1}^m A'_j \otimes A'_j + A''_j \otimes A''_j = (PF)^{t_2} = R((PF)^{t_2}) = uu^t + \sum_{j=1}^m (a'_j)(a'_j)^t + (a''_j)(a''_j)^t$, where $R(Id) = uu^t$, $R(A'_j \otimes A'_j) = V(A'_j)V(A'_j)^t = (a'_j)(a'_j)^t$, $R(A''_j \otimes A''_j) = V(A''_j)V(A''_j)^t = (a''_j)(a''_j)^t$. Notice that $V(A'_j) = a'_j \in \text{Skew-Sym}(\mathbb{C}^n \otimes \mathbb{C}^n)$ and $V(A''_j) = a''_j \in \text{Skew-Sym}(\mathbb{C}^n \otimes \mathbb{C}^n)$, for every j .

Thus, $2Id - uu^t - (\sum_{j=1}^m (a'_j)(a'_j)^t + (a''_j)(a''_j)^t) = Id - (\sum_{j=1}^m A'_j \otimes A'_j + A''_j \otimes A''_j) = PF$.

Finally, since $a'_j, a''_j \in \text{Skew-Sym}(\mathbb{C}^n \otimes \mathbb{C}^n)$ then $(a'_j)^t u = (a''_j)^t u = 0$ and $PFu = (2 - u^t u)u$. Now, $u^t u = n$ and n is the tensor rank of w . Since PF is a positive semidefinite symmetric matrix then the tensor rank of w is smaller or equal to 2.

Recall that $B = w\bar{w}^t + \sum_{j=1}^m b_j \bar{b}_j^t$ is PPT, $w \in \text{Sym}(\mathbb{C}^k \otimes \mathbb{C}^k)$ and $b_j \in \text{Skew-Sym}(\mathbb{C}^k \otimes \mathbb{C}^k)$, $1 \leq j \leq m$.

Now, if the tensor rank of w is 2 then $r = v_1 \otimes v_1 + v_2 \otimes v_2$, such that v_1 and v_2 are linear independent. Let $M \in M_k$ be such that $\ker(M) = \text{span}\{v_1 + iv_2\}$. Notice that $(M \otimes M)w = 0$.

Next, $(M \otimes M)B(M^* \otimes M^*) = \sum_{j=1}^m c_j \bar{c}_j^t$, where $c_j = (M \otimes M)b_j \in \text{Skew-Sym}(\mathbb{C}^k \otimes \mathbb{C}^k)$.

Notice that $(M \otimes M)B(M^* \otimes M^*)$ is PPT, therefore $0 \leq \text{tr}(((M \otimes M)B(M^* \otimes M^*))^{t_2} v \bar{v}^t) = \text{tr}((\sum_{j=1}^m c_j \bar{c}_j^t)^{t_2} v v^t) = \text{tr}(\sum_{j=1}^m c_j \bar{c}_j^t (v v^t)^{t_2})$, for every $v \in \mathbb{C}^k \otimes \mathbb{C}^k$. If we choose $v_0 = \sum_{i=1}^k f_i \otimes f_i$, where $\{f_1, \dots, f_k\}$ is the canonical basis of \mathbb{C}^k then $(v_0 \bar{v}_0^t)^{t_2} = F$. So $0 \leq \text{tr}((\sum_{j=1}^m c_j \bar{c}_j^t)^{t_2} v_0 \bar{v}_0^t) = \text{tr}(\sum_{j=1}^m c_j \bar{c}_j^t F) = -\text{tr}(\sum_{j=1}^m c_j \bar{c}_j^t) \leq 0$, since $\bar{c}_j^t F = -\bar{c}_j^t$. Thus, $\text{tr}(\sum_{j=1}^m c_j \bar{c}_j^t) = 0$ and every $c_j = 0$.

Thus, every $b_j \in \ker(M \otimes M) \cap \text{Skew-Sym}(\mathbb{C}^k \otimes \mathbb{C}^k) = \text{Skew-Sym}((v_1 + iv_2) \otimes \mathbb{C}^k + \mathbb{C}^k \otimes (v_1 + iv_2))$.

Now, let $M' \in M_k$ be such that $\ker(M') = \text{span}\{v_1 - iv_2\}$. Notice that $(M' \otimes M')w = 0$. We can repeat the argument above using M' instead of M . So $b_j \in \text{Skew-Sym}((v_1 - iv_2) \otimes \mathbb{C}^k + \mathbb{C}^k \otimes (v_1 - iv_2))$.

Hence, every $b_j \in \text{Skew-Sym}((v_1 + iv_2) \otimes \mathbb{C}^k + \mathbb{C}^k \otimes (v_1 + iv_2)) \cap \text{Skew-Sym}((v_1 - iv_2) \otimes \mathbb{C}^k + \mathbb{C}^k \otimes (v_1 - iv_2)) = \text{span}\{(v_1 + iv_2) \otimes (v_1 - iv_2) - (v_1 - iv_2) \otimes (v_1 + iv_2)\} = \text{span}\{v_1 \otimes v_2 - v_2 \otimes v_1\}$. Therefore, $(Id - F)\rho(I - F) = \sum_{j=1}^m b_j \bar{b}_j^t = s \bar{s}^t$, where $s \in \text{span}\{v_1 \otimes v_2 - v_2 \otimes v_1\}$.

If the tensor rank of w is 1 then $w = v_1 \otimes v_1$. Let $N \in M_k$ be such that $\ker(N) = \text{span}\{v_1\}$. Notice that $(N \otimes N)r = 0$. We can repeat the argument above to conclude that $b_j \in \ker(N \otimes N) \cap \text{Skew-Sym}(\mathbb{C}^k \otimes \mathbb{C}^k) = \text{Skew-Sym}(v_1 \otimes \mathbb{C}^k + \mathbb{C}^k \otimes v_1)$.

Next, if there is $l \in \{1, \dots, m\}$ such that $0 \neq b_l$ then there exists $v_3 \in \mathbb{C}^k$ such that $b_l = v_1 \otimes v_3 - v_3 \otimes v_1$ and v_1, v_3 are linear independent. Let $O \in M_{2 \times k}$ be such that $Ov_1 = g_1$, $Ov_3 = g_2$, where $\{g_1, g_2\}$ is the canonical basis of \mathbb{C}^2 . Thus, $(O \otimes O)r = g_1 \otimes g_1$, $(O \otimes O)b_l = g_1 \otimes g_2 - g_2 \otimes g_1$ and $(O \otimes O)b_j \in \text{Skew-Sym}(\mathbb{C}^2 \otimes \mathbb{C}^2) = \text{span}\{g_1 \otimes g_2 - g_2 \otimes g_1\}$, for $1 \leq j \leq m$. Therefore, the image of $(O \otimes O)B(O^* \otimes O^*)$ is generated by $g_1 \otimes g_1$ and $g_1 \otimes g_2 - g_2 \otimes g_1$. So the only tensor with tensor rank 1 in this image is $g_1 \otimes g_1$ and $(O \otimes O)B(O^* \otimes O^*) \in M_2 \otimes M_2$ is not separable by the range criterion (see [2]). This is a contradiction, since $(O \otimes O)B(O^* \otimes O^*)$ is PPT and in $M_2 \otimes M_2$ every PPT matrix is separable (see [13]). Therefore, every $b_j = 0$ and $(Id - F)A(Id - F) = 0$.

Finally, in both cases ($B = w \bar{w}^t + s \bar{s}^t$ or $B = v_1 \bar{v}_1^t \otimes v_1 \bar{v}_1^t$) the ranges of the marginal matrices of $B = 2(\rho + F\rho F)$ are subspaces of $\text{span}\{v_1, v_2\}$. Thus, the marginal rank of $\rho + F\rho F$ (its marginal matrices are equal) is smaller or equal to 2. Hence, the marginal ranks of ρ are also smaller or equal to 2. Since ρ is PPT then ρ is separable, by Horodecki theorem (see [13]). \square

Corollary 3.6. *Let $\rho \in M_k \otimes M_k \simeq M_{k^2}$ be a positive semidefinite hermitian matrix, $Id \in M_k \otimes M_k \simeq M_{k^2}$ the identity and $F \in M_k \otimes M_k$ the flip operator. If ρ is positive under partial transposition, r is the marginal rank of $\rho + F\rho F$ and $r \leq 3$ then $\text{rank}(Id + F)\rho(Id + F) \geq \max\{\frac{2}{r}\text{rank}(Id - F)\rho(Id - F), \frac{r}{2}\}$.*

Proof. If $r \leq 2$ then the marginal ranks of ρ are also smaller or equal to 2. Since ρ is PPT then ρ is separable, by Horodecki theorem (see [13]). The theorem follows by theorem 2.4.

Now, if $r = 3$ then $\text{rank}(Id + F)\rho(Id + F) \geq 2$, by theorem 3.5. Since $\text{rank}((Id - F)\rho(Id - F)) = \text{rank}((Id - F)(\rho + F\rho F)(Id - F)) \leq \dim(\text{Skew-Sym}(\mathbb{C}^3 \otimes \mathbb{C}^3)) = \frac{3 \times 2}{2} = 3$ then $\text{rank}((Id + F)\rho(Id + F)) \geq \max\{\frac{2}{3}\text{rank}((Id - F)\rho(Id - F)), \frac{3}{2}\}$. \square

4. SPC MATRICES

Let r be the marginal rank of $\rho + F\rho F$. Here, we provide several examples of PPT matrices $\rho \in M_k \otimes M_k$ such that $\text{rank}((Id + F)\rho(Id + F)) \geq r \geq \frac{2}{r-1}\text{rank}((Id - F)\rho(Id - F))$ (see corollary 4.6). So it is not trivial to find PPT entanglement in the gap discussed in the begining of the last section. The main result of this section is the following: If ρ is PPT and $\rho + F\rho F$ is symmetric with positive coefficients (definition 4.2) then $\text{rank}((Id + F)\rho(Id + F)) \geq r \geq \frac{2}{r-1}\text{rank}((Id - F)\rho(Id - F))$ (theorem 4.4).

Definition 4.1. A decomposition of a matrix $A \in M_k \otimes M_m$, $\sum_{i=1}^n \lambda_i \gamma_i \otimes \delta_i$, is a Schmidt decomposition if $\{\gamma_i \mid 1 \leq i \leq n\} \subset M_k$, $\{\delta_i \mid 1 \leq i \leq n\} \subset M_m$ are orthonormal sets with respect to the trace inner product, $\lambda_i \in \mathbb{R}$ and $\lambda_i > 0$. Also, if γ_i and δ_i are Hermitian matrices for every i , then $\sum_{i=1}^n \lambda_i \gamma_i \otimes \delta_i$ is a Hermitian Schmidt decomposition of A .

Definition 4.2. (SPC matrices) Let $A \in M_k \otimes M_k \simeq M_{k^2}$ be a positive semidefinite Hermitian matrix. We say that A is symmetric with positive coefficients or simply SPC, if A has the following symmetric Hermitian Schmidt decomposition with positive coefficients: $\sum_{i=1}^n \lambda_i \gamma_i \otimes \gamma_i$, with $\lambda_i > 0$, for every i .

Remark 4.3. The following description of SPC matrices can be found in [11, Corollary 25]: $A \in M_k \otimes M_k \simeq M_{k^2}$ is SPC if and only if A and $R(A^{t_2})$ are positive semidefinite Hermitian matrices.

Theorem 4.4. If $\rho \in M_k \otimes M_k \simeq M_{k^2}$ is a PPT matrix, $\rho + F\rho F$ is a SPC matrix and r is the marginal rank of $\rho + F\rho F$ then $(Id + F)\rho(Id + F)$ is also a PPT matrix and $\text{rank}((Id + F)\rho(Id + F)) \geq r \geq \frac{2}{r-1} \text{rank}((Id - F)\rho(Id - F))$.

Proof. Since ρ is a positive semidefinite Hermitian matrix then $F\rho F$ and $\rho + F\rho F$ are too. Let $C = \rho + F\rho F$.

Notice that $C = \frac{1}{2}(Id + F)\rho(Id + F) + \frac{1}{2}(Id - F)\rho(Id - F)$. Let $\xi = \frac{1}{2}(Id + F)\rho(Id + F)$ and $\eta = \frac{1}{2}(Id - F)\rho(Id - F)$. Observe that ξ and η are positive semidefinite Hermitian matrices.

Now, $F\xi F = \xi$, $F\eta F = \eta$, therefore $\xi^B = (F\xi F)^A = \xi^A$ and $\eta^B = (F\eta F)^A = \eta^A$, by remark 2.3. Thus, $\xi^A + \eta^A = C^A = C^B = \rho^A + \rho^B$.

Observe that if $v \in \ker(C^A)$ then $v \in \ker(\xi^A)$, since ξ^A, η^A are positive semidefinite (see remark 2.3). Therefore, $\text{rank}(\xi^A) \leq \text{rank}(C^A)$.

Next, if $v \in \ker(\xi^A)$ then $0 = \text{tr}(\xi^A v \bar{v}^t) = \text{tr}(\xi(v \bar{v}^t \otimes Id))$. Thus, $\text{tr}(\xi(v \bar{v}^t \otimes v \bar{v}^t)) = 0$, since ξ is positive semidefinite.

Since $v \otimes v \in \text{Sym}(\mathbb{C}^k \otimes \mathbb{C}^k) \subset \ker(\eta)$ then $\text{tr}(\eta(v \bar{v}^t \otimes v \bar{v}^t)) = 0$. So $\text{tr}(C(v \bar{v}^t \otimes v \bar{v}^t)) = 0$.

By hypothesis, C is a SPC matrix, therefore $C = \sum_{i=1}^m \lambda_i \gamma_i \otimes \gamma_i$, where γ_i is Hermitian and $\lambda_i > 0$, for $1 \leq i \leq m$ (see definition 4.2). Thus, $\sum_{i=1}^m \lambda_i \text{tr}(\gamma_i v \bar{v}^t)^2 = 0$ and $\text{tr}(\gamma_i v \bar{v}^t) = 0$, for $1 \leq i \leq m$.

Since C^A is positive semidefinite, $C^A = \sum_{i=1}^m \lambda_i \text{tr}(\gamma_i) \gamma_i$ and $\text{tr}(C^A v \bar{v}^t) = 0$ then $v \in \ker(C^A)$. Therefore, $\text{rank}(C^A) \leq \text{rank}(\xi^A)$ and $\text{rank}(C^A) = \text{rank}(\xi^A)$.

Now, let us prove that ξ is PPT. Since $\xi F = \xi$ and $\eta F = -\eta$ then $2\xi = C + CF$. Thus, $2\xi^{t_2} = C^{t_2} + (CF)^{t_2} = C^{t_2} + R(C^{t_2})$, by item 5 in lemma 3.3. Now, C^{t_2} is positive semidefinite, since ρ and $F\rho F$ are PPT, and $R(C^{t_2})$ is positive semidefinite because C is SPC, by remark 4.3.

Finally, by [7, Theorem 1], since ξ is PPT then $\text{rank}(\xi) \geq \text{rank}(\xi^A) = \text{rank}(C^A) = r$. By remark 2.3, $\Im(C) \subset \Im(C^A) \otimes \Im(C^B) = \Im(C^A) \otimes \Im(C^A)$. Thus, $\Im(\eta) \subset \Im(C) \cap \text{Skew-Sym}(\mathbb{C}^k \otimes \mathbb{C}^k) \subset \text{Skew-Sym}(\Im(C^A) \otimes \Im(C^A))$. Therefore, $\text{rank}(\eta) \leq \frac{r(r-1)}{2}$ and $\text{rank}(\xi) \geq r \geq \frac{2}{r-1} \text{rank}(\eta)$. \square

Remark 4.5. The next two examples show that the hypothesis, $\rho + F\rho F$ is SPC, cannot be dropped in theorem 4.4. The first example is the separable matrix $\rho \in M_k \otimes M_k$ of corollary 2.7, which satisfies $\text{rank}((Id + F)\rho(Id + F)) = \frac{2}{k} \text{rank}((Id - F)\rho(Id - F)) < \frac{2}{k-1} \text{rank}((Id - F)\rho(Id - F))$. The second example is the matrix $\rho = B + C$, where $B = k(\sum_{i=1}^k e_i e_i^t \otimes e_i e_i^t) - uu^t$, $u = \sum_{i=1}^k e_i \otimes e_i$, $\{e_1, \dots, e_k\}$ is the canonical basis of \mathbb{C}^k and $C = Id - F$. This matrix ρ is a positive semidefinite Hermitian matrix and invariant under partial transposition, since the partial transposition of F is uu^t , and vice versa, therefore ρ is PPT. Notice that $\text{rank}((Id + F)\rho(Id + F)) = \text{rank}(B) = k - 1$ and $\text{rank}((Id - F)\rho(Id - F)) = \text{rank}(C) = \frac{k(k-1)}{2}$. Thus, $\text{rank}((Id + F)\rho(Id + F)) = \frac{2}{k} \text{rank}((Id - F)\rho(Id - F)) < \frac{2}{k-1} \text{rank}((Id - F)\rho(Id - F))$.

Corollary 4.6. *Let $\rho \in M_k \otimes M_k \simeq M_{k^2}$ be a PPT and SPC matrix then $\text{rank}((Id+F)\rho(Id+F)) \geq r \geq \frac{2}{r-1} \text{rank}((Id-F)\rho(Id-F))$, where r is the marginal rank of $\rho + F\rho F$.*

Proof. Since ρ is SPC then $\rho = \sum_{i=1}^m \lambda_i \gamma_i \otimes \gamma_i$, by definition 4.2. Thus, $F\rho F = \rho$ and $\rho + F\rho F = 2\rho$ is SPC. Now, use theorem 4.4. \square

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